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On an identity in law between Brownian quadratic functionals

Ju-Yi Yen [a,∗], Marc Yor [b,c]

[a] Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221-0025, USA
[b] Institut Universitaire de France, Ministère de l'Enseignement Supérieur et de la Recherche Scientifique, France
[c] Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, Paris, France

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ABSTRACT

A stochastic Fubini argument and a computation of some moments are given in relation to a distributional integration by parts formula for Brownian quadratic functionals.

1. Introduction

Let $(B_t, t \geq 0)$ denote a one-dimensional Brownian motion starting from 0. Let $0 \leq a \leq b$. Consider also $f, g : [a, b] \to \mathbb{R}_+$, two continuous functions, with f decreasing, and g increasing (in the large, following the "European convention"). It was shown in Mansuy and Yor (2008) that the following identity in law between quadratic functionals of Brownian motion holds:

$$\int_a^b -B^2_{g(x)} df(x) + f(b)B^2_{g(b)} \overset{\text{(law)}}{=} g(a)B^2_{f(a)} + \int_a^b B^2_{f(x)} dg(x) \tag{1}$$

To make things simple, we assume that both f and g are C^1, and that $f(b) = g(a) = 0$. Thus (1) can be written as

$$\int_a^b -f'(x)\, B^2_{g(x)} dx \overset{\text{(law)}}{=} \int_a^b g'(x) B^2_{f(x)} dx \tag{2}$$

which is equivalent to

$$\int_a^b -f'(x)(B^2_{g(x)} - g(x))dx \overset{\text{(law)}}{=} \int_a^b g'(x)(B^2_{f(x)} - f(x))dx. \tag{3}$$

In this paper,

- we prove (2), using a kind of stochastic Fubini argument—see Section 2;
- we prove that the moments of order 2 and 3 of the two sides of (3) are equal, but the complexity of the computations limited us to these orders—see Section 3.

However, we still think that, hidden behind (3), there is a sequence of remarkable integration by parts formulae which would explain the equality of the moments on the two sides of (3).

∗ Corresponding author. Tel.: +1 6152981759.
E-mail address: ju-yi.yen@uc.edu (J.-Y. Yen).

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2. A stochastic Fubini argument

To prove (2), we write the Laplace transform

$$I = E\left[\exp\left(-\frac{\lambda^2}{2}\int_a^b dx|f'(x)|B_{g(x)}^2\right)\right]$$

as the characteristic function

$$E\left[\exp\left(i\lambda\int_a^b dC_x\sqrt{|f'(x)|}B_{g(x)}\right)\right]$$

where (C_x) is a second Brownian motion independent of B.

Next, we remark that

$$B_{g(x)} \overset{\text{(law)}}{=} \int_a^x \sqrt{g'(y)}dB_y \quad \text{(as processes)}$$

and we use Fubini's argument:

$$\int_a^b dC_x\sqrt{|f'(x)|}\int_a^x \sqrt{g'(y)}dB_y = \int_a^b \sqrt{g'(y)}dB_y\int_y^b dC_x\sqrt{|f'(x)|}.$$

We then remark that

$$\int_y^b dC_x\sqrt{|f'(x)|} \overset{\text{(law)}}{=} C_{f(y)} \quad \text{(as processes)}$$

and so

$$I = E\left[\exp\left(-\frac{\lambda^2}{2}\int_a^b g'(y)dy\, C_{f(y)}^2\right)\right].$$

Comparing these two expressions for I, we are led to

$$\int_a^b |f'(x)|B_{g(x)}^2 dx \overset{\text{(law)}}{=} \int_a^b g'(y)\, C_{f(y)}^2 dy$$

which is (2), since $B \overset{\text{(law)}}{=} C$.

3. Moments of order 2 and 3 on the two sides of (3)

3.1

We introduce the notation $M_u = B_u^2 - u$, and $\bar{\beta}_n(u_1, \ldots, u_n) = E[M_{u_1}, \ldots, M_{u_n}]$ $(u_1 < u_2 < \cdots < u_n)$. Let $\bar{L}_n$ and $\bar{R}_n$ denote the nth moments of the LHS and RHS respectively of (3). It is easily shown that

$$\begin{aligned}\bar{L}_n &= n!\int_a^b |f'(y_1)|dy_1\int_{y_1}^b |f'(y_2)|dy_2\cdots\int_{y_{n-1}}^b |f'(y_n)|dy_n\,\bar{\beta}_n(g(y_1), \ldots, g(y_n))\\ \bar{R}_n &= n!\int_a^b g'(x_1)dx_1\int_a^{x_1} g'(x_2)dx_2\cdots\int_a^{x_{n-1}} g'(x_n)dx_n\,\bar{\beta}_n(f(x_1), \ldots, f(x_n)).\end{aligned} \tag{4}$$

In the following subsections, we shall show the equality of $\bar{L}_n$ and $\bar{R}_n$ for $n = 2$ and $n = 3$, with the help of the formulae

$$\begin{aligned}\bar{\beta}_2(u_1, u_2) &= 2u_1^2\\ \bar{\beta}_3(u_1, u_2, u_3) &= 8u_1^2u_2\end{aligned} \tag{5}$$

which may be obtained from standard stochastic calculus. We have also computed

$$\bar{\beta}_4(u_1, u_2, u_3, u_4) = 24u_1^2u_2^2 + 32u_1^2u_2u_3 + 4u_1^2u_3^2$$

but we shall not use this formula.