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Parametric estimation for the scale parameter for scale distributions using moving extremes ranked set sampling

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ABSTRACT

A modification of ranked set sampling (RSS) called moving extremes ranked set sampling (MERSS) is considered for the estimation of the scale parameter of scale distributions. A maximum likelihood estimator (MLE) is studied and its properties are obtained. We prove the MLE is an equivariant estimator under scale transformation. In order to give more insight into the performance of MERSS with respect to (w.r.t.) simple random sampling (SRS), the asymptotic efficiency of the MLE using MERSS w.r.t. that using SRS is computed for some usual scale distributions. The relative results show that the MLE using MERSS can be real competitors to the MLE using SRS, when the same sample size is used.

1. Introduction

Ranked set sampling (RSS) was introduced by McIntyre (1952) for estimating the pasture yields. It is appropriate for situations where quantification of sampling units is either costly or difficult, but ranking the units in a small set is easy and inexpensive. Ranking can be performed based on expert judgment, visual inspection or any means that does not involve actually quantifying the observations. In RSS one first draws m^2 units at random from the population and partitions them into m sets of m units. The m units in each set are ranked without making actual measurements. From the first set of m units the unit ranked lowest is chosen for actual quantification. From the second set of m units the unit ranked second lowest is measured. The process is continued until the unit ranked largest is measured from the m-th set of m units. If a larger sample size is required then the procedure can be repeated r times to obtain a sample of size $n = rm$. These chosen elements are called a ranked set sample.

Takahasi and Wakimoto (1968) established a very important statistical foundation for the theory of RSS. For more research work on parametric methods for RSS, see Fei et al. (1994), Lam et al. (1994) and Samawi et al. (1996). However, ranking accuracy affects the efficiency of the estimator. When the set size is large, ranking error tends to occur. In order to reduce the error of ranking and keep optimality inherited in the original RSS procedure, Al-Odat and Al-Saleh (2001) introduced the concept of varied set size RSS, which is coined here as Moving Extremes Ranked Set Sampling (MERSS).

The procedure of MERSS is described as follows:

1. Select m simple random sample sets of size $1, 2, 3, \ldots, m$, respectively.
2. Order the elements of each set by visual inspection or other relatively inexpensive methods, without actual measurement of the characteristic of interest.
3. Measure accurately the maximum ordered observation from the first set, then the second set, …, the last set.
4. Step (3) is repeated on another m sets of size $1, 2, \ldots, m$ respectively, however the minimum ordered observations are measured instead of the maximum ordered observations.
5. Repeat the above steps r times until the desired sample size, $n = 2rm$ is obtained.

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Clearly, only the two extreme values are used in MERSS, maximum or minimum of sets of varied size, whereas the ranks of all the elements of each set are needed in RSS. Since it is not difficult to identify maximum or minimum units, MERSS is a very useful modification of RSS. It allows for an increase of set size without introducing too many ranking errors.

Al-Saleh and Al-Hadhrami (2003a) studied MLE of location distributions based on MERSS. They demonstrated that MLE of MERSS is always performed better than simple random sampling (SRS) for the case of normal distribution. Al-Saleh and Al-Hadhrami (2003b) studied the MLE and its properties under exponential distribution based on MERSS and they showed that MLE of MERSS is always performed better than SRS numerically.

In this paper MLE of scale distributions is studied based on MERSS. The MLE of the scale parameter is derived in Section 2. The properties of the MLE are stated in Section 3. It shows the MLE is an equivariant estimator under scale transformation. Existence of the MLE for some usual scale distributions is demonstrated in Section 4. In order to give more insight into the performance of MERSS with respect to (w.r.t.) SRS, we study the asymptotic efficiency of the MLE using MERSS w.r.t. that using SRS. Since under some regularity conditions, the asymptotic efficiency of the MLE can be obtained from the inverse of the Fisher information number, we compute the Fisher information number for scale parameter in Section 5. The Fisher information numbers of some usual scale distributions are given in this section. The asymptotic efficiency of the MLE using MERSS w.r.t. that using SRS is compared numerically for these scale distributions in Section 6. The relative results show that the MLE using MERSS is always more efficient than the MLE using SRS, when the same sample size is used.

2. The maximum likelihood estimator

Let $\{X_{i1}^1, X_{i2}^1, \ldots\ldots, X_{ii}^1\}$ and $\{Y_{i1}^1, Y_{i2}^1, \ldots\ldots, Y_{ii}^1\}$ $i = 1, \ldots, m$ be $2m$ sets of random samples from X, and they are independent. X is a scale distribution random variable with probability density function (pdf) $f(x, \sigma)$, i.e., $f(x, \sigma) = \frac{1}{\sigma} g(\frac{x}{\sigma})$, where $g(\frac{x}{\sigma})$ is known. Denote $X_{ii} = \max\{X_{i1}^1, X_{i2}^1, \ldots\ldots, X_{ii}^1\}$ and $Y_{1i} = \min\{Y_{i1}^1, Y_{i2}^1, \ldots\ldots, Y_{ii}^1\}$, $i = 1, \ldots, m$. Then $\{X_{mm}, X_{m-1m-1}, \ldots\ldots, X_{11}, Y_{1m}, Y_{1m-1}, \ldots\ldots, Y_{11}\}$ is a MERSS from X. Note that the elements of this sample are independent of each other. If judgment ranking is accurate, then X_{ii} has the same distribution as the ith order statistics (maximum) of a SRS of size i from $f(x, \sigma)$, i.e. the pdf of X_{ii} is

$$f_{ii}(x, \sigma) = i\frac{1}{\sigma} g\left(\frac{x}{\sigma}\right)\left[G\left(\frac{x}{\sigma}\right)\right]^{i-1},$$

where $G(\frac{x}{\sigma})$ is the distribution function.

Also Y_{1i} has the same distribution as the 1st order statistics (minimum) of a SRS of size i from $f(x, \sigma)$, i.e. the pdf of Y_{1i} is

$$f_{1i}(x, \sigma) = i\frac{1}{\sigma} g\left(\frac{x}{\sigma}\right)\left[1 - G\left(\frac{x}{\sigma}\right)\right]^{i-1}.$$

In order to get the MLE of σ we start with the likelihood function:

$$L(\sigma) = \prod_{i=1}^{m} i\frac{1}{\sigma} g\left(\frac{x_{ii}}{\sigma}\right)\left[G\left(\frac{x_{ii}}{\sigma}\right)\right]^{i-1} i\frac{1}{\sigma} g\left(\frac{y_{1i}}{\sigma}\right)\left[1 - G\left(\frac{y_{1i}}{\sigma}\right)\right]^{i-1}.$$

The log likelihood is

$$L^*(\sigma) = C + \sum_{i=1}^{m} \log g\left(\frac{x_{ii}}{\sigma}\right) + \sum_{i=1}^{m} \log g\left(\frac{y_{1i}}{\sigma}\right) + \sum_{i=1}^{m}(i-1)\log G\left(\frac{x_{ii}}{\sigma}\right) + \sum_{i=1}^{m}(i-1)\log\left(1 - G\left(\frac{y_{1i}}{\sigma}\right)\right) + \log\frac{1}{\sigma^{2m}},$$

where C is a constant.

Taking the first derivative for $L^*(\sigma)$, we have

$$\frac{\partial L^*}{\partial(\sigma)} = -\frac{1}{\sigma^2}\sum_{i=1}^{m}\frac{g'\left(\frac{x_{ii}}{\sigma}\right)x_{ii}}{g\left(\frac{x_{ii}}{\sigma}\right)} - \frac{1}{\sigma^2}\sum_{i=1}^{m}\frac{g'\left(\frac{y_{1i}}{\sigma}\right)y_{1i}}{g\left(\frac{y_{1i}}{\sigma}\right)} - \frac{1}{\sigma^2}\sum_{i=1}^{m}(i-1)\frac{g\left(\frac{x_{ii}}{\sigma}\right)x_{ii}}{G\left(\frac{x_{ii}}{\sigma}\right)} + \frac{1}{\sigma^2}\sum_{i=1}^{m}(i-1)\frac{g\left(\frac{y_{1i}}{\sigma}\right)y_{1i}}{\left(1 - G\left(\frac{x_{ii}}{\sigma}\right)\right)} - \frac{2m}{\sigma}. \quad (1)$$

Under the regularity assumptions (see Lehmann (1983, p. 440–441)), if the MLE of σ exists, then it is a solution of $\frac{\partial L^*}{\partial(\sigma)} = 0$. From Eq. (1), we get the MLE of σ denoted by $\hat{\sigma}_{\text{MERSS}}$, which satisfies

$$\sum_{i=1}^{m}\frac{g'\left(\frac{x_{ii}}{\sigma}\right)x_{ii}}{g\left(\frac{x_{ii}}{\sigma}\right)} + \sum_{i=1}^{m}\frac{g'\left(\frac{y_{1i}}{\sigma}\right)y_{1i}}{g\left(\frac{y_{1i}}{\sigma}\right)} + \sum_{i=1}^{m}(i-1)\frac{g\left(\frac{x_{ii}}{\sigma}\right)x_{ii}}{G\left(\frac{x_{ii}}{\sigma}\right)} - \sum_{i=1}^{m}(i-1)\frac{g\left(\frac{y_{1i}}{\sigma}\right)y_{1i}}{\left(1 - G\left(\frac{x_{ii}}{\sigma}\right)\right)} + 2m\sigma = 0. \quad (2)$$