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Finding hitting times in various graphs

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ABSTRACT

The hitting time, h_{uv}, of a random walk on a finite graph G, is the expected time for the walk to reach vertex v given that it started at vertex u. We present two methods of calculating the hitting time between vertices of finite graphs, along with applications to specific classes of graphs, including grids, trees, and the 'tadpole' graphs.

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1. Introduction

A random walk on a graph is a walk that begins at a particular starting vertex in which each successive step is determined by randomly choosing an edge adjacent to the previous vertex and traveling to the vertex at the other endpoint of the edge. This random choice is distributed equally over all edges adjacent to the vertex. For our purposes, we will consider only random walks on unweighted and undirected graphs. Although much of the historical work on random walks considers infinite graphs, recent work has dealt more with finite graphs. When dealing with random walks on finite graphs, the focus turns to less qualitative questions; rather than asking whether or not a random walk will return to its starting vertex, it may be interesting to ask how long the random walk would take to return to the starting vertex.

A property that arises from analyzing random walks on finite graphs is the hitting time. Given a finite graph G, the hitting time, h_{uv}, from a vertex u to a vertex v, is the expected number of steps it takes for a random walk that starts at vertex u to reach vertex v. Note that when dealing with finite graphs, the hitting time from u to v is finite if and only if the vertices u and v are connected.

The hitting time of a random walk has many useful properties. For example, the cover time, the expected time it takes for a random walk to visit all vertices of a graph, can be both bounded above by a function of the largest hitting time from one vertex to another, and below by a function of the smallest hitting time from one vertex to another (Lovász, 1993). However, the hitting times between vertices of various graphs can be hard to analyze, and finding their values is not intuitive. There exist a few known bounds on hitting times, as found in Brightwell and Winkler (1990), Cogill and Peng (2010) and Palacios (1990, 1994). We will focus on finding exact formulas for the hitting time for certain types of graphs.

We will consider two different methods of calculating the hitting time of a graph and demonstrate their applications to various classes of graphs. Section 2 gives an explicit formula for finding the hitting time, but only in a few specific cases. Specifically, the formula can only be used to find the hitting time from one vertex to a neighbor and only if the graph exhibits a symmetry about the starting vertex. This can then be applied to random walks on a variety of classes of graphs, including grids, hypercubes, and trees.

Section 3 uses the method of calculating hitting times through a system of linear equations first shown in Palacios (1990). Although this is not as convenient as a formula, this method can be applied to random walks on any graph. We can use this

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to derive formulas for hitting times of random walks on graphs for which the method of Section 2 does not apply, such as for the complete d-ary tree, or the tadpole graph.

2. Hitting times in graphs with symmetry

2.1. Proof of theorem

The following theorem gives a formula that can, in certain cases, be used to find hitting times.

Theorem 2.1. *Let v be a vertex of a connected graph G with neighbor u. If for every other neighbor of v, there exists an automorphism of G that maps u to that neighbor of v, then the hitting time from u to v is $\frac{2e}{k} - 1$, where e is the number of the edges in the graph, and k is the number of neighbors of v.*

Proof. Because of the symmetry of the graph, the hitting time is equal from any neighbor of v to v. Let this value be x.

Now consider the random walk on G that starts at vertex v and moves to vertex u in the next step, and the next such t in which the tth vertex visited is v and the $(t+1)$th vertex visited is u. We can find the expected value of the next such t, both in terms of both k and x, and in terms of e, allowing us to solve for x. This can also be thought of as finding the recurrence time of a random walk along the edges of the directed graph G', formed by replacing each edge in G with two, one in each direction.

Consider the structure of the walk more closely. After starting at v and then going to u, in order to again return back to v and then u, the walk must first return to v. This takes an expected x steps. At this point, the walk can continue in two different ways; the walk may go to u, with a probability of $\frac{1}{k}$ and 1 additional step. It may also continue on to another vertex, in which case the walk must again return back to v. This adds on average, another $x+1$ to the number of steps the random walk has taken. Continuing in this manner, we find that the expected recurrence time is

$$\frac{1}{k}(x+1) + \frac{1}{k}\frac{k-1}{k}2(x+1) + \frac{1}{k}\frac{k-1}{k}\frac{k-1}{k}3(x+1) + \cdots$$

which simplifies to $k(x+1)$.

It is well known that the expected recurrence time in this walk is $2e$, as there are $2e$ edges in G' (Lovász, 1993). Setting these two values equal and solving for x gives $x = \frac{2e}{k} - 1$. □

2.2. Applications to the grid, hypercube, and trees

This technique can be used to find the hitting time in the following graphs.

Corollary 2.2. *In a d-dimensional grid whose dimensions all have length m, the hitting time from a corner to one of its neighbors is $2\,(m-1)\,m^{d-1} - 1$.*

Proof. There are $d\,(m-1)\,m^{d-1}$ edges in a d-dimensional grid whose dimensions all have length m. Applying Theorem 2.1 gives the hitting time given above. □

In particular, the application of this technique to the hypercube gives a more general result, as every vertex is a corner.

Corollary 2.3. *In a d-dimensional hypercube, the hitting time from any vertex to one of its neighbors is $2^d - 1$.*

Additionally, this technique can be used to find the hitting times from one vertex to a neighbor in trees.

Corollary 2.4. *The hitting time from a vertex v to a neighbor u in a tree is $2e - 1$, where e is the number of edges in the connected component containing u after the edges to all other neighbors of u are removed.*

Proof. Removing the edges to all other neighbors of u does not change the hitting time from v to u, as it is not possible for a random walk starting at v to reach those vertices without having already reached u. However, this allows the hypothesis of Theorem 2.1 to hold for the connected component containing u, as u now only has 1 neighbor. Finally, we can just apply the result to this modified graph. □

3. Hitting time via a system of linear equations

The second technique we will use to find hitting times can be applied to any graph, rather than graphs with some property of symmetry, but does not yield formulas as easily. The following theorem, first shown in Palacios (1990) gives a set of linear equations whose solution gives the hitting times of a random walk on a graph. If we know the general structure of the graph, then it may be possible to find the hitting times in terms of certain properties of the graph.

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