Statistics and Probability Letters 83 (2013) 2113–2118

Contents lists available at SciVerse ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Inequalities involving expectations to characterize distributions

Subarna Bhattacharjee[a], Asok K. Nanda[b,*], Satya Kr. Misra[c]

[a] Department of Mathematics, Ravenshaw University, Cuttack-753003, Odisha, India

[b] Department of Mathematics and Statistics, IISER Kolkata, Mohanpur Campus, Mohanpur-741252, West Bengal, India

[c] School of Applied Sciences, KIIT University, Bhubaneswar-751024, Odisha, India

ARTICLE INFO

Article history:
Received 8 January 2013
Received in revised form 20 May 2013
Accepted 20 May 2013
Available online 27 May 2013

Keywords:
Gini's mean difference
Logistic distribution
Log-odds rate

ABSTRACT

The characterization of distributions is well known in the field of Statistics and Reliability. This paper characterizes a few distributions with the help of failure rate, mean residual, log-odds rate, and aging intensity functions.

1. Introduction

We mention a few functions to be used in the sequel from the vast literature of reliability theory. These functions characterize the aging phenomenon of any living unit or a system of components.

Suppose that X is a continuous random variable with probability density function (pdf) $f(\cdot)$, cumulative distributive function (cdf) $F(\cdot)$ and survival function (sf) $\bar{F}(\cdot) \equiv 1 - F(\cdot)$. The failure rate function of X, denoted by $r(\cdot)$, is defined as the ratio of the pdf to the sf, i.e., $r(x) = \frac{f(x)}{\bar{F}(x)}$, where defined. The mean residual life function, denoted by $m(\cdot)$ of X, is defined as $m(x) = E(X - x \mid X > x) = \frac{\int_x^\infty \bar{F}(u)du}{\bar{F}(x)}$. The aging intensity function, which analyzes the aging property of a system quantitatively (cf. Jiang et al., 2003), is defined as

$$\begin{aligned} L(x) &= \frac{xr(x)}{\int_0^x r(u)du}, \quad \text{where defined,} \\ &= \frac{-xf(x)}{\bar{F}(x)\ln \bar{F}(x)}. \end{aligned}$$

The log-odds rate (LOR) of X is defined as

$$\begin{aligned} LOR_X(x) &= \frac{d}{dx} LO_X(x), \\ &= \frac{f(x)}{F(x)\,\bar{F}(x)} \end{aligned} \tag{1.1}$$

* Corresponding author. Tel.: +91 9474722103.
E-mail addresses: asoknanda@yahoo.com, asok.k.nanda@gmail.com, asok@iiserkol.ac.in (A.K. Nanda).

0167-7152/$ – see front matter
http://dx.doi.org/10.1016/j.spl.2013.05.022

where $LO_X(\cdot) = \ln \frac{F(\cdot)}{\bar{F}(\cdot)}$ is the log-odds function. Also, by changing the variable $Y = \ln(X)$, the log-odds rate in terms of $\ln(x)$ is obtained as

$$LOR_Y(y) = \frac{g(y)}{G(y)\,\bar{G}(y)}, \\ = \frac{e^y f(e^y)}{F(e^y)\,\bar{F}(e^y)}, \tag{1.2}$$

where the random variable Y has the pdf and the cdf denoted by $g(\cdot)$ and $G(\cdot)$ respectively. Henceforth, we denote $LOR_X(x)$ by $LOR(x)$ and $LOR_Y(y)$ by $LOR(y)$ when there is no ambiguity.

The odds ratio has a large number of applications in different fields viz. reliability, large sample theory, discriminant analysis and many others. The usefulness of the log-odds rate function in comparing the reliability of two systems is discussed in Navarro et al. (2008). They have characterized different probability distributions based on the relationships among conditional moment, failure rate and log-odds rate. It has been used to characterize probability distributions by Sunoj et al. (2007), where the authors also mention the different usage of the log-odds rate in reliability and repairable systems. Characterization of distribution by the log-odds rate is also discussed in Wang et al. (2003). They have noted that the increasing log-odds ratio is less stringent than increasing failure rate, and is therefore potentially of broader applicability. Brown et al. (2012) have used the log-odds ratio to construct large sample Wilson-type confidence intervals. It has been numerically demonstrated by Platt (1998) that the asymptotic bias of the maximum modified profile likelihood estimator of a common odds ratio is negligible for odds ratio less than 5. While estimating discriminant coefficients, Sheena and Gupta (2004) obtained the estimators in terms of gradient of log-odds. Fisher's linear discriminant function can be viewed as a posterior log-odds that a subject belongs to one population versus the other given the data vector, see, for instance, Haff (1986). He has also shown that the vector of discriminant coefficients is the gradient of the posterior log-odds. Zimmer et al. (1998) and Wang et al. (2003) showed that F has constant LOR in x (resp. $\ln x$) if and only if F has logistic (resp. log–logistic) distribution with respective cdf given by

$$F_1(x) = \frac{1}{1+\exp\left(-\frac{x-\mu}{s}\right)}, \quad x \in R, \mu \in R, s > 0,$$

and

$$F_2(x) = \frac{x^\alpha}{1+x^\alpha}, \quad x > 0, \alpha > 0.$$

The characterization of probability distributions arising in reliability theory is done in Kagan et al. (1973), Kotz (1974), Galambos (1975a,b), Klebanov (1978), Nanda (2010) among others. The characterization of distributions through truncation is done in Laurent (1974). Looking into the importance of the log-odds ratio, here we characterize a few well-known statistical distributions through $r(\cdot)$, $m(\cdot)$, $L(\cdot)$ and $LOR(\cdot)$.

2. Main results

In this section, we characterize a few probability distributions viz., exponential, Weibull, logistic and log–logistic distributions. We start this section by stating one known result from Makino (1984) and Nanda (2010).

Theorem 2.1. *For any nonnegative random variable X,*

$$E\left(\frac{1}{r(X)}\right) \geq \frac{1}{E(r(X))},$$

and

$$E\left(\frac{1}{m(X)}\right) \geq \frac{1}{E(m(X))}.$$

The equality holds if and only if X is exponentially distributed.

This motivates us to prove the following theorem. Before that we give a lemma from Nanda et al. (2007) to be used in sequel.

Lemma 2.1. *For a nonnegative random variable X, $L(x) = c$, for $x \geq 0$, c being a constant, if and only if X follows two-parameter Weibull distribution with shape parameter c.* □

Theorem 2.2. *For any nonnegative random variable X,*

$$E\left(\frac{1}{L(X)}\right) \geq \frac{1}{E(L(X))}.$$

The equality holds if and only if X follows two-parameter Weibull family of distributions.