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A local limit theorem for densities of the additive component of a finite Markov Additive Process

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ABSTRACT

In this paper, we are concerned with centered Markov Additive Processes $\{(X_t, Y_t)\}_{t\in\mathbb{T}}$ where the driving Markov process $\{X_t\}_{t\in\mathbb{T}}$ has a finite state space. Under suitable conditions, we provide a local limit theorem for the density of the absolutely continuous part of the probability distribution of $t^{-1/2}Y_t$ given X_0. The rate of convergence and the moment condition are the expected ones with respect to the i.i.d case. An application to the joint distribution of local times of a finite jump process is sketched.

1. Introduction

When $\{X_k\}_{k\geq 1}$ is a sequence of centered independent and identically distributed (i.i.d.) real valued random variables such that $Y_n := \sum_{i=1}^n X_i$ has a bounded density for some n, it is well-known that the density f_n of $n^{-1/2}Y_n$ satisfies the following Local Limit Theorem (LLT)

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}} |f_n(y) - \eta(y)| = 0,$$

where $\eta(\cdot)$ is the density of the Gaussian distribution $\mathcal{N}(0, \sigma^2)$, with $\sigma^2 := \mathbb{E}[X_1^2]$; see Gnedenko (1954), Ibragimov and Linnik (1971), and Feller (1971) for detailed discussions. If X_1 has a bounded density and a third moment, then the rate of the previous convergence is $O(n^{-1/2})$; see Siraždinov and Šahaĭdarova (1965), Šahaĭdarova (1966), and Korolev and Zhukov (1998).

This paper extends the last result to centered Markov Additive Processes (MAPs) $\{(X_t, Y_t)\}_{t\in\mathbb{T}}$ with state space $\mathbb{X} \times \mathbb{R}^d$, where $\mathbb{X} := \{1, \ldots, N\}$ and $\mathbb{T} := \mathbb{N}$ or $\mathbb{T} := [0, \infty)$. Recall from Asmussen (2003) that $\{(X_t, Y_t)\}_{t\in\mathbb{T}}$ is a Markov process on $\mathbb{X} \times \mathbb{R}^d$ with a transition semi-group, denoted by $\{Q_t\}_{t\in\mathbb{T}}$, which satisfies

$$\forall (k, y) \in \mathbb{X} \times \mathbb{R}^d,\ \forall (\ell, B) \in \mathbb{X} \times B(\mathbb{R}^d), \quad Q_t(k, y; \{\ell\} \times B) = Q_t(k, 0; \{\ell\} \times B - y). \tag{1}$$

The transition semi-group of the driving Markov process $\{X_t\}_{t\in\mathbb{T}}$ is denoted by $\{P_t\}_{t\in\mathbb{T}}$. The stochastic $N \times N$-matrix $P := P_1$ is assumed to be irreducible and aperiodic. Moreover the mass of the singular part of the conditional probability distribution of $t^{-1/2}Y_t$ given $X_0 = k$ is supposed to converge exponentially fast to zero. Let $f_{k,t}(\cdot)$ be the density of the absolutely continuous part of this conditional distribution. Under a third moment condition on Y_t and some conditions on $f_{k,t}(\cdot)$ and its Fourier transform (see (AC1)–(AC2)), we prove in Theorem 2.2 that, for every $k \in \mathbb{X}$, the density $f_{k,t}(\cdot)$ satisfies essentially the

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following property:

$$\sup_{y\in\mathbb{R}^d} |f_{k,t}(y) - \eta_\Sigma(y)| = O\left(t^{-1/2}\right)$$

where $\eta_\Sigma(\cdot)$ denotes the density of $\mathcal{N}(0, \Sigma)$. Matrix Σ is the asymptotic covariance provided by the Central Limit Theorem (CLT) and is assumed to be invertible. Our moment condition and rate of convergence are the expected ones with respect to the i.i.d. case. The proof of Theorem 2.2 is based on the spectral method (e.g. see Guivarc'h and Hardy, 1988; Hennion and Hervé, 2001 when $\mathbb{T} := \mathbb{N}$, and Ferré et al., 2012 when $\mathbb{T} := [0, +\infty)$).

To the best of our knowledge, Theorem 2.2 is new. The known contribution to LLT for densities of additive components of MAP is in Rahimzadeh Sani (2008), where only the discrete time case is considered and exponential-type moment condition on Y_1 is assumed (the rate of convergence is not addressed). Note that, for discrete time, our Theorem 2.2 only requires a third moment condition on Y_1. Moreover it is worth noticing that the probability distribution of Y_t is not assumed to be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. An application to the joint distribution of local times of a finite jump process is provided in Section 3. The proof of Theorem 2.2 is given in the last section. In order to save space, some details are reported in a companion paper which is referred to as (HL).[1]

This work was motivated by a question in relation with a self-attracting continuous time process called the Vertex Reinforced Jump Process (VRJP) recently investigated by Sabot and Tarrès (2011). This process is closely related to the Edge Reinforced Random Walk (ERRW) and was instrumental in the proof of the recurrence of the ERRW in all dimensions at strong reinforcement. In a paper in progress, Sabot and Tarrès (in preparation) make a link between the VRJP and accurate pointwise large deviation for reversible Markov jump processes: it appears that the limit measure of the VRJP (Sabot and Tarrès, 2011) is closely related to the first order of pointwise large deviations which are derived in Sabot and Tarrès (in preparation) for continuous time Markov processes using the present local limit theorem and Remark 2.3.

Notations. Any vector $v \equiv (v_k) \in \mathbb{C}^N$ is considered as a row-vector and $v^\top$ is the corresponding column-vector. The vector with all components equal to 1 is denoted by $\mathbf{1}$. The Euclidean scalar product and its associated norm on $\mathbb{C}^N$ is denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. The set of $N \times N$-matrices with complex entries is denoted by $\mathcal{M}_N(\mathbb{C})$. We use the following norm $\|\cdot\|_0$ on $\mathcal{M}_N(\mathbb{C})$:

$$\forall A \equiv (A_{k,\ell}) \in \mathcal{M}_N(\mathbb{C}), \quad \|A\|_0 := \max\left\{|A_{k,\ell}| : (k,\ell) \in \{1,\ldots,N\}^2\right\}.$$

For any bounded positive measure ν on $\mathbb{R}^d$, we define its Fourier transform as:

$$\forall \zeta \in \mathbb{R}^d, \quad \widehat{\nu}(\zeta) := \int_{\mathbb{R}^d} e^{i\langle\zeta,y\rangle}\, d\nu(y).$$

Let $\mathcal{A} \equiv (\mathcal{A}_{k,\ell})$ be a $N \times N$-matrix with entries in the set of bounded positive measures on $\mathbb{R}^d$. We set

$$\forall B \in B(\mathbb{R}^d), \quad \mathcal{A}(1_B) := \left(\mathcal{A}_{k,\ell}(1_B)\right), \qquad \forall \zeta \in \mathbb{R}^d, \quad \widehat{\mathcal{A}}(\zeta) := (\widehat{\mathcal{A}}_{k,\ell}(\zeta)). \tag{2}$$

2. The LLT for the density process

Let $\{(X_t, Z_t)\}_{t\in\mathbb{T}}$ be an MAP with state space $\mathbb{X} \times \mathbb{R}^d$, where $\mathbb{X} := \{1,\ldots,N\}$ and the driving Markov process $\{X_t\}_{t\in\mathbb{T}}$ has transition semi-group $\{P_t\}_{t\in\mathbb{T}}$. We refer to Asmussen (2003, Chapter XI) for the basic material on such MAPs. The conditional probability to $\{X_0 = k\}$ and its associated expectation are denoted by $\mathbb{P}_k$ and $\mathbb{E}_k$ respectively. Note that if $T : \mathbb{R}^d \to \mathbb{R}^m$ is a linear transformation, then $\{X_t, T(Z_t)\}_{t\in\mathbb{T}}$ is still a MAP on $\mathbb{X} \times R^m$ (see Lemma C.1 in (HL)). We suppose that $\{X_t\}_{t\in\mathbb{T}}$ has a unique invariant probability measure π. Set $m := \mathbb{E}_\pi[Z_1] = \sum_k \pi(k)\mathbb{E}_k[Z_1] \in \mathbb{R}^d$. Consider the centered MAP $\{(X_t, Y_t)\}_{t\in\mathbb{T}}$ where $Y_t := Z_t - t\,m$. The two next assumptions are involved in both CLT and LLT below.

(I-A): *The stochastic $N \times N$-matrix $P := P_1$ is irreducible and aperiodic.*
(Mα): *The family of r.v. $\{Y_v\}_{v\in(0,1]\cap\mathbb{T}}$ satisfies the uniform moment condition of order α:*

$$M_\alpha := \max_{k\in\mathbb{X}} \sup_{v\in(0,1]\cap\mathbb{T}} \mathbb{E}_k\left[\|Y_v\|^\alpha\right] < \infty. \tag{3}$$

The next theorem provides a CLT for $t^{-1/2}Y_t$, proved when $d := 1$ in Keilson and Wishart (1964) for $\mathbb{T} = \mathbb{N}$ and in Fukushima and Hitsuda (1967) for $\mathbb{T} = [0, \infty)$; see Ferré et al. (2012) for ρ-mixing driving Markov processes.

Theorem 2.1. *Under assumptions* (I-A) *and* (M2), $\{t^{-1/2}Y_t\}_{t\in\mathbb{T}}$ *converges in distribution to a d-dimensional Gaussian law* $\mathcal{N}(0, \Sigma)$ *when* $t \to +\infty$.

[1] Available at http://arxiv.org/abs/1305.5644.