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An almost sure limit theorem for the maxima of smooth stationary Gaussian processes☆

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ABSTRACT

Let $\{X(t), t \geq 0\}$ be a continuous mean square differentiable stationary Gaussian process. Under some mild restrictions on its correlation function $r(\cdot)$, we prove an almost sure limit theorem for the maximum of the Gaussian process $\{X(t), t \geq 0\}$.

1. Introduction

The asymptotic theory of Gaussian processes has experienced new developments in the recent past years. Significant recent contributions can be found in Adler et al. (in press) and Cheng and Xiao (2013) for smooth or continuous Gaussian random fields. Exact asymptotic type results derived for extremes of non-smooth Gaussian processes by relying on the Double Sum Method are given in Arendarczyk and Dębicki (2011, 2012), Dębicki and Tabiś (2011), Hüsler et al. (2011), and Tan et al. (2012). Along with the analysis of extremes of continuous maxima, novel results inspired by the seminal paper Piterbarg (2004) have been derived for the joint asymptotic behaviour of discrete and continuous maxima; see Tan and Hashorva (2013, submitted for publication). Yet another important direction which deals with extremes of Gaussian processes followed in this paper concerns the almost sure limit theorem (ASLT).

The ASLT has been first introduced independently by Brosamler (1988) and Schatte (1988) for partial sum, and then the concept has already started to receive applications in many areas. For example, Bercu (2004) has showed the statistical applications of ASLT. In its simplest form the ASLT for maxima $M_n = \max_{k\leq n} X_k$ of independent random variables $X_i, i \geq 1$ states that (see Cheng et al., 1998; Fahrner and Stadtmüller, 1998) under some regularity conditions

$$\lim_{N\to\infty} \frac{1}{\ln N} \sum_{n=1}^{N} \frac{1}{n} I(a_n(M_n - b_n) \leq x) = G(x) \quad \text{a.s.} \tag{1}$$

for some real sequences $a_n > 0, b_n \in \mathbb{R}, n \geq 1$ and some non-degenerate distribution G. Throughout this paper $I(\cdot)$ stands for the indicator function.

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Csáki and Gonchigdanzan (2002) have extended (1) for weakly dependent stationary Gaussian sequences; see also Tan and Peng (2009), Peng and Nadarajah (2011), Weng et al. (2012), and Hashorva and Weng (2013). For the stationary and non-stationary Gaussian random fields, we refer the reader to Choi (2010) and Tan and Wang (in press), respectively.

In this paper, we are interested in similar problems for the maxima of stationary Gaussian processes.

Let $\{X(t), t \geq 0\}$ be a continuous mean square differentiable stationary Gaussian process, assumed to have zero mean and covariance function $r(t) = EX(s)X(t+s)$ satisfying the following condition

$$r(t) = 1 - \frac{\lambda}{2}|t|^2 + o(|t|^2) \quad \text{as } t \to 0, \tag{2}$$

where $\lambda = -r''(0)$. Note that condition (2) is one of the standard conditions for studying the extremes for Gaussian processes; see e.g., Leadbetter et al. (1983) and Piterbarg (2004). Next, set $M(T) = M([0,T]) = \max\{X(t), 0 \leq t \leq T\}$ and let $N_u(T)$ be the number of upcrossings of the level u by $\{X(t), 0 \leq t \leq T\}$, so that by Rice's formula (see e.g. Leadbetter et al., 1983, p. 153)

$$\mu = \mu(u) = EN_u(1) = \frac{1}{2\pi}\lambda^{1/2}e^{-u^2/2}. \tag{3}$$

Under suitable conditions on the rate of decay of $r(t)$, e.g. if $r(t)\log t \to 0$ as $t \to \infty$ and if u and T tend to infinity in a coordinated way, $EN_u(T) = T\mu(u) \to \tau$ for some constant $\tau \geq 0$, then as $T \to \infty$ (c.f. Leadbetter et al., 1983, Chapters 8 and 9)

$$P(M(T) \leq u) \to e^{-\tau} \tag{4}$$

and

$$P(a_T(M(T) - b_T) \leq u) \to \exp\left(-e^{-x}\right) \tag{5}$$

where the normalizing constants are defined for all large T by

$$a_T = \sqrt{2\ln T}, \quad b_T = a_T + a_T^{-1}\ln\left(\frac{\lambda^{1/2}}{2\pi}\right). \tag{6}$$

The above result can be extended to the more general case of nondifferentiable Gaussian processes; see Chapter 12 of Leadbetter et al. (1983) for details. In this note, we concentrate on the ASLT for the maxima of stationary Gaussian processes. The obtained result is a continuous version of the one obtained by Csáki and Gonchigdanzan (2002) and is of theoretical importance since it is a stronger version of the "usual" convergence result.

The brief organization of the paper is as follows. In Section 2 we give the main result while in Section 3, we prove the main result.

2. Main result

In addition to the notation and hypotheses from the Introduction we will assume that

$$r''(t) - r''(0) \leq ct^2, \quad t \geq 0, \tag{7}$$

for some constant $c > 0$. This assumption is not a severe one, it is satisfied for instance when $r(t) = \exp(-t^2/2)$ and $r(t) = 1/(1+t^2/4)^2$; see Kratz and Rootzén (1997) for details. Next, we state the main result of this contribution.

Theorem 2.1. *Let $\{X(t), t \geq 0\}$ be a continuous mean square differentiable stationary Gaussian process with correlation functions $r(\cdot)$ satisfying (2) and (7). Suppose in addition that $r(T)(\ln T)(\ln\ln T)^{3(1+\varepsilon)} = O(1)$ for some $\varepsilon > 0$. Then,*

(i) *if $T\mu(u_T) \to \tau$ for $0 < \tau < \infty$, then*

$$\lim_{T\to\infty}\frac{1}{\ln T}\int_1^T \frac{1}{t} I\left(\max_{1\leq s\leq t} X(s) \leq u_t\right)dt = e^{-\tau} \quad a.s., \tag{8}$$

(ii) *if a_T, b_T are defined as in (6), then*

$$\lim_{T\to\infty}\frac{1}{\ln T}\int_1^T \frac{1}{t} I\left(a_t\left(\max_{1\leq s\leq t} X(s) - b_t\right) \leq x\right)dt = \exp\left(-e^{-x}\right) \quad a.s. \tag{9}$$

Remark 2.1. Assumption (7) is from Kratz and Rootzén (1997). In that paper, the authors dealt with the rate of convergence of extremes for the mean square differentiable stationary Gaussian processes. To prove the ASLT for extremes of Gaussian processes, we also need to know a similar type of convergence rate.

Remark 2.2. The above mentioned two examples $r(t) = \exp(-t^2/2)$ and $r(t) = 1/(1+t^2/4)^2$ also satisfy the conditions of Theorem 2.1. The first example is the type of squared exponential covariance function and the second example is the type of rational quadratic covariance function.