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An Osgood criterion for integral equations with applications to stochastic differential equations with an additive noise

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ABSTRACT

In this paper we use a comparison theorem for integral equations to show that the classical Osgood criterion can be applied to solutions of integral equations of the form

$$X_t = a + \int_0^t b(X_s)ds + g(t), \quad t \geq 0.$$

Here, g is a measurable function such that

$$\limsup_{t\to\infty} \left(\inf_{0\leq h\leq 1} g(t+h) \right) = \infty,$$

and b is a positive and non-decreasing function. Namely, we will see that the solution X explodes in finite time if and only if $\int_{\cdot}^{\infty} \frac{ds}{b(s)} < \infty$. As an example, we use the law of the iterated logarithm to see that the bifractional Brownian motion and some increasing self-similar Markov processes satisfy the above condition on g. In other words, g can represent the paths of these processes.

1. Introduction

Let $b, \sigma : \mathbb{R} \to \mathbb{R}$ be two measurable functions and $W = \{W_t : t \geq 0\}$ a Brownian motion defined on a complete probability space. The Feller test allows us to see if the solution of the stochastic differential equation (SDE)

$$\begin{aligned} dY_t &= b(Y_t)dt + \sigma(Y_t)dW_t, \quad t > 0, \\ Y_0 &= a, \end{aligned} \tag{1.1}$$

explodes in finite time knowing only the coefficients b and σ (see Proposition 5.1). That is, the paths of the solution Y may go to infinity if the parameter t approaches some finite random time. So, roughly speaking, the explosion time of this equation is defined as $T_e := \sup\{t > 0 : |Y_t| < \infty\}$.

In the case that, in Eq. (1.1), the diffusion term is zero (i.e., $\sigma = 0$) and the drift b is positive, the phenomenon of explosion for ordinary differential equations (ODE) is well known. In fact, let υ be the solution of

$$\begin{aligned} d\upsilon(t) &= b(\upsilon(t))dt, \quad t > 0, \\ \upsilon(0) &= a. \end{aligned} \tag{1.2}$$

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Then, Osgood condition (Osgood, 1898) establishes that $T_e < \infty$ if and only if

$$\int_{\cdot}^{\infty} \frac{\mathrm{d}s}{b(s)} < \infty. \tag{1.3}$$

Moreover, for this equation, we have $T_e = \int_a^\infty \frac{\mathrm{d}s}{b(s)}$. In order to get an idea of the proof of this fact, the reader can see Section 2. However, the precise instant of the explosion of the solution of the SDE (1.1) is not easy to figure out, and some numerical schemes have been improved to approximate the time of explosion (see Dávila et al., 2005).

The comparison theorem for SDEs has been an important tool to analyze either the explosion in finite time, or the existence of global solutions. Indeed, this tool allows us to find lower and upper bounds of the solution of the equation under consideration, which are solutions of ODE of the form (1.2), or similar equations. Thus, the problem reduces to verify an integral condition similar to condition (1.3), as we do in this paper, and as it is done, for example, by Groisman and Rossi (2007, see Example 2.1) and by Ikeda and Watanabe (1977). Also, we can associate stochastic partial differential equations with ODE via comparison theorems (see Pérez and Villa, 2010a,b), which increases the fields of applications of blow-up of ODE. Therefore, the study of the explosion in finite time of ODE of the form (1.2) is of great interest and they have applications in chemical reactions, molecular biology, hydrodynamics, heat conduction, models of anomalous growth, etc. (see Pérez and Villa, 2010a,b, and references therein). The reader can also see de Pablo et al. (2005) (and references therein) for a nice survey on the problem and applications of explosion of reaction–diffusion equations.

The Feller test for the blow-up in finite time of Eq. (1.1) is equivalent to condition (1.3) if σ is a constant function and b is a positive and non-decreasing function, as it is shown in Section 5. The main tool for the Feller test is the Itô's formula (see Karatzas and Shreve, 1991). Consequently, we cannot use this test neither for Eq. (1.2) when it is perturbed by an additive noise that is not a semimartingale, as, for instance, the bifractional Brownian motion, nor for the perturbed equation

$$Y_t' = \phi(t)h(Y_t) + \psi(t)z(Y_t), \quad t > 0, \tag{1.4}$$

where $\phi, \psi, h, z : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions.

Constantin (1995) has studied sufficient conditions for the existence of global solutions to the perturbed equation (1.4). Among these conditions we can indicate either

$$\limsup_{t\to\infty} \frac{z(t)}{h(t)} < \infty, \quad \text{or} \quad \liminf_{t\to\infty} \frac{z(t)}{h(t)} > 0. \tag{1.5}$$

In this paper we utilize a comparison theorem for integral equations to state that the explosion in finite time of the solution to

$$X_t = a + \int_0^t b(X_s)\mathrm{d}s + g(t), \quad t \geq 0, \tag{1.6}$$

is equivalent to Osgood condition (1.3). Here, g is a measurable function such that

$$\limsup_{t\to\infty} \left(\inf_{0\leq h\leq 1} g(t+h) \right) = \infty,$$

and b is a positive and non-decreasing function (see Hypotheses (H1) and (H2)). As a consequence, this result is also true when we change g by a process $Z = \{Z_t : t \geq 0\}$ whose paths are such that

$$\limsup_{t\to\infty} \left(\inf_{0\leq h\leq 1} Z_{t+h} \right) = \infty, \quad \text{a.s.} \tag{1.7}$$

It is worth mentioning that similar conditions on b have been considered in a generalization of Osgood condition (1.3) obtained by Mydlarczyk and Okrasiński (2000), and in Dávila et al. (2005), Constantin (1995, Corollary 2.2) and Groisman and Rossi (2007, Example 2.1), among others.

Also, in this paper we use the law of the iterated logarithm to show that condition (1.7) holds for some processes. Namely, the bifractional Brownian motion and some self-similar Markov processes. Observe that conditions (1.5) are similar to the law of iterated logarithm. Motivated from their properties, self-similar processes have been considered as an input noise in hydrology, telecommunications, mathematical finance and queueing theory (see Mandelbrot and van Ness, 1968). In particular, the fractional Brownian motion (fBm) is a zero-mean Gaussian process whose covariance of its increments on intervals decays asymptotically as a negative power of the distance between the intervals. FBm is the only finite-variance process which is self-similar and has stationary increments. This process has been generalized by the bifractional Brownian motion, which is also a Gaussian process, has some properties of fBm (for instance, self-similarity and stationarity for small increments, see (4.2)), and enlarges the modelling tool kit.

The existence and uniqueness of a global solution to Eq. (1.6) has been studied by Nualart and Ouknine (2002, 2003) under different suitable conditions on b, when g represents the paths of the fractional Brownian motion.

The paper is organized as follows. Our comparison theorem is given in Section 2. The main result is stated and proved in Section 3. That is, the blow-up in finite time for Eq. (1.6). In Section 4 we introduce some examples of processes satisfying (1.7). Finally, the relation between (1.3) and Feller test for Eq. (1.1) is established in Section 5.