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The strong law of large numbers for dependent random variables

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Abstract

This paper establishes two results for the strong law of large numbers under negative association and ϱ-mixing. In their proofs, a Háyek–Rényi-type maximal inequality is employed.

MSC: primary 60F15; 60G50

Keywords: Strong law of large numbers; Negatively associated random variables; ϱ-mixing sequence; Háyek–Rényi-type maximal inequality

1. Introduction

Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables defined on the probability space $(\Omega, \mathscr{F}, P)$ and let $S_n = \sum_{i=1}^{n} X_i$.

In this paper, we consider sufficient conditions for the strong law of large numbers to hold. The results presented in this paper are obtained by using the maximal inequality of Háyek–Rényi-type (Háyek and Rényi (1955)). Fazekas and Klesov (2000) proved the following strong law of large numbers by using this inequality.

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Theorem 1. *Let* $\{b_n, n \geqslant 1\}$ *be a nondecreasing, unbounded sequence of positive numbers. Let* $\{\alpha_n, n \geqslant 1\}$ *be nonnegative numbers. Let r be a fixed positive number. Assume that for each* $n \geqslant 1$,

$$E\left[\max_{1 \leqslant i \leqslant n} |S_i|\right]^r \leqslant \sum_{i=1}^{n} \alpha_i. \tag{1}$$

If

$$\sum_{i=1}^{\infty} \frac{\alpha_i}{b_i^r} < \infty, \tag{2}$$

then

$$\lim_{n \to \infty} \frac{S_n}{b_n} = 0 \quad a.s.\ n \longrightarrow \infty. \tag{3}$$

Using this theorem, we are going to show that Kolmogorov's, Chung's (1947) and Teicher's (1968) strong law of large numbers for independent random variables $\{X_n, n \geqslant 1\}$ can be generalized to the case of negatively associated random variables. In the second part of this paper, we consider a ϱ-mixing sequences $\{X_n, n \geqslant 1\}$. Fazekas and Klesov (2000) obtained the strong law of large numbers for the ϱ-mixing sequences $\{X_n, n \geqslant 1\}$ by using Theorem 1, under the assumption $E|X_n|^q < \infty$ for $n \geqslant 1$, and $q \geqslant 2$. The result presented in this paper allows us to extend the result of Fazekas and Klesov to ϱ-mixing sequences $\{X_n, n \geqslant 1\}$ for which the moments of order q, $q > 2$, do not exist.

Now we present the definition of negative association due to Esary et al. (1967), and Joag-Dev and Proschan (1983).

Definition 1. The random variables $X_1, X_2, \ldots, X_n$ are said to be negatively associated if for any disjoint subset $A, B \subset \{1, 2, \ldots, n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^A$ and g on $\mathbb{R}^B$,

$$\mathrm{cov}(f(X_k, k \in A), g(X_k, k \in B)) \leqslant 0 \tag{4}$$

provided the covariance exists.

An infinite sequence $\{X_n, n \geqslant 1\}$ of random variable is said to be negatively associated if every finite subset $\{X_{i_1}, X_{i_2}, \ldots, X_{i_k}\}$ is a set of negatively associated random variables.

Some results for sums of negatively associated random variables we can find in Matuła (1992).

2. The strong law of large numbers for negatively associated random variables

Theorem 2. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of negatively associated random variables and let* $\varphi : \mathbb{R} \to \mathbb{R}^+$ *be an even, continuous and nondecreasing on* $(0, \infty)$ *function with* $\lim_{x \to \infty} \varphi(x) = \infty$, *and such that*

(a) $\varphi(x)/x \searrow$ *or*
(b) $\varphi(x)/x \nearrow$ *and* $\varphi(x)/x^p \searrow$, $x \to \infty$ *for some* $1 < p \leqslant 2$.