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Statistics & Probability Letters 73 (2005) 71–78
STATISTICS & PROBABILITY LETTERS
www.elsevier.com/locate/stapro

A note on prediction and interpolation errors in time series

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Received 28 July 2004
Available online 8 April 2005

Abstract

In this note, we analyze the relationship between one-step ahead prediction errors and interpolation errors in time series. We obtain an expression of the prediction errors in terms of the interpolation errors and then we show that minimizing the sum of squares of the one-step ahead standardized prediction errors is equivalent to minimizing the sum of squares of standardized interpolation errors.

Keywords: Fixed-point smoothing; Interpolation error; Kalman filter; Prediction error

1. Introduction

It is well known that the likelihood function of an ARMA(p, q) process can be written in terms of the one-step ahead prediction errors using the conditional distribution of each observation given the previous data. This is called the prediction error decomposition. The maximum likelihood estimate (MLE) of the parameters can be computed by minimizing the concentrated likelihood function, which depends on the one-step ahead prediction errors. The interpolation problem consists in the estimation of a missing observation by using the past and future values of the time series. The interpolator which minimizes the mean-squared error criterion is computed by the expected value of the observation given the rest of the sample. The interpolation error is the difference between the interpolated value and the true value of the observation. In the state-space form of ARMA models, the interpolator is obtained with some smoothing algorithm, such as the

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0167-7152/$ - see front matter
doi:10.1016/j.spl.2005.03.014

fixed point smoothing (FPS) (see Anderson and Moore, 1979). The aim of this note is to show the relationship between prediction errors and interpolation errors and to prove that the parameter values which minimize the mean squared prediction error are the same as those which minimize the mean-squared interpolation errors. This note is organized as follows. In Section 2 we introduce the notation and briefly review the FPS algorithm. In Section 3, we first obtain an expression of the one-step ahead prediction error in terms of the interpolation errors, second we derive the covariances between interpolation errors and third we show that minimizing the sum of squares of the one-step ahead standardized prediction errors leads to the same result as minimizing the sum of squares of the standardized interpolation errors. Section 4 illustrates the result in the simplest case of a first-order autoregressive process.

2. Kalman filter and fixed point smoothing

Let $\{z_t\}$ be a process following a zero mean stationary and invertible ARMA(p,q) model,

$$\phi(B)z_t = \theta(B)u_t, \tag{1}$$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$, $\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$ and $\{u_t\}$ is a sequence of independent $N(0,\sigma^2)$ variables. We denote the vector of ARMA parameters in (1) by $\beta = (\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q)'$ and a sample generated by this process by $z = (z_1, \ldots, z_T)'$, where T is the sample size. Let Σ_z be the covariance matrix of z, then the likelihood function is

$$L(z|\beta,\sigma^2) = (2\pi)^{-T/2}|\Sigma_z|^{-1/2} \exp\left(-\frac{z'\Sigma_z^{-1}z}{2}\right).$$

Let $z_{t|t-1} = E[z_t|z_{t-1}, \ldots, z_1]$ for $t = 1, \ldots, T$, be the one-step ahead predictions obtained by minimizing the mean squared errors, where $z_{1|0} = E[z_1]$, and let $e_t = z_t - z_{t|t-1}$ be the corresponding one-step ahead prediction errors with variances $E[(z_t - z_{t|t-1})^2] = \sigma^2 v_{t|t-1}^2$, and where $var(z_1) = \sigma^2 v_{1|0}^2$. The log-likelihood, $\ell(z|\beta,\sigma^2) = \log L(z|\beta,\sigma^2)$, can be written as

$$\ell(z|\beta,\sigma^2) = -\frac{T}{2}\log 2\pi\sigma^2 - \frac{1}{2}\sum_{t=1}^{T} \log v_{t|t-1}^2 - \frac{1}{2\sigma^2}\sum_{t=1}^{T} \frac{e_t^2}{v_{t|t-1}^2}$$

and the MLE of σ^2 is given by

$$\widehat{\sigma}^2_{\mathrm{MLE}} = \frac{1}{T}\sum_{t=1}^{T} \frac{e_t^2}{v_{t|t-1}^2} \tag{2}$$

and, using (2), the MLE of β, $\widehat{\beta}_{\mathrm{MLE}}$, maximizes the concentrated log-likelihood given by

$$S(\beta) = \frac{1}{T}\sum_{t=1}^{T} \log v_{t|t-1}^2 + \log\left(\sum_{t=1}^{T} \frac{e_t^2}{v_{t|t-1}^2}\right).$$