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# Biases of the maximum likelihood and Cohen–Sackrowitz estimators for the tree-order model ☆

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**Abstract**

Consider $s+1$ univariate normal populations with common variance $\sigma^2$ and means $\mu_i$, $i=0,1,\dots,s$, constrained by the tree-order restrictions $\mu_i \geqslant \mu_0$, $i=1,2,\dots,s$. For certain sequences $\mu_0,\mu_1,\dots$ the maximum likelihood-based estimator (MLBE) of $\mu_0$ diverges to $-\infty$ as $s\to\infty$ and its bias is unbounded. By contrast, the bias of an alternative estimator of $\mu_0$ proposed by Cohen and Sackrowitz (J. Statist. Plan. Infer. 107 (2002) 89–101) remains bounded. In this note the biases of the MLBEs of the other components $\mu_1,\mu_2,\dots$ are studied and compared to the biases of the corresponding Cohen–Sackrowitz estimators (CSE). Unlike the MLBE of $\mu_0$, the MLBEs of $\mu_i$ for $i\geqslant 1$, are asymptotically unbiased in most cases. By contrast, the CSEs of $\mu_i$, $i=1,2,\dots,s$ more often have nonzero asymptotic bias.


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*Keywords:* Order-restricted inference; Tree order; Maximum likelihood estimator; Cohen–Sackrowitz estimator; Bias

## 1. Introduction

Let $X_0, X_1, \dots, X_s$ be independent normally distributed random variables with $X_i \sim \mathrm{N}(\mu_i, \sigma^2)$, where the means $\mu_0, \mu_1, \dots, \mu_s$ satisfy the tree-order restriction $\mu_i \geqslant \mu_0$, $i=1,\dots,s$. This restriction

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commonly occurs when comparing one treatment to several controls. For notational simplicity, assume that $\sigma^2$ is known, say $\sigma^2 = 1$, and that the sample sizes $n_0 = n_1 = \cdots = n_s = 1$. Then the MLE $\hat{\mu}$ of $\mu \equiv (\mu_0, \mu_1, \ldots, \mu_s)$ is given by (cf. Lee, 1988; Brunk, 1965; Barlow et al., 1972; Robertson et al., 1988)

$$\hat{\mu}_0 = \min_{S \subseteq \{1,2,\ldots,s\}} \left\{ \frac{X_0 + \sum_{j \in S} X_j}{1 + |S|} \right\}, \tag{1}$$

$$\hat{\mu}_i = \max(\hat{\mu}_0, X_i), \quad i = 1, \ldots, s, \tag{2}$$

from which it is seen that $\hat{\mu}$ is *translation-equivariant*, i.e.

$$\hat{\mu}(X_0 + a, X_1 + a, \ldots, X_s + a) = \hat{\mu}(X_0, X_1, \ldots, X_s) + (a, a, \ldots, a). \tag{3}$$

It follows from (3) and the translation equivariance of the normal distribution that the bias $\hat{b}(\mu)$ of $\hat{\mu}$ is translation-invariant, i.e.

$$\hat{b}(\mu) \equiv E_\mu(\hat{\mu}) - \mu = \hat{b}(\mu + (a, a, \ldots, a)) \quad \forall a \in \mathbb{R}. \tag{4}$$

Set $a = -\mu_0$ to obtain

$$\hat{b}(\mu) = \hat{b}(0, \mu_1 - \mu_0, \ldots, \mu_s - \mu_0). \tag{5}$$

Similarly, the Cohen–Sackrowitz estimator $\tilde{\mu} \equiv \tilde{\mu}^{(s)}$ (cf. Section 3) is translation equivariant and its bias is translation-invariant. Therefore, to study and compare the biases of $\hat{\mu}$ and $\tilde{\mu}$, we may assume hereafter that $\mu_0 = 0$ and $\mu_i \geqslant 0$ for $i = 1, \ldots, s$.

From (1), $\hat{\mu}_0 \leqslant X_0$ and the inequality is strict with positive probability, so $E_\mu(\hat{\mu}_0) < E_\mu(X_0) \equiv \mu_0 = 0$, hence $\hat{\mu}_0$ is negatively biased. Theorem 2.1 of Lee (1988) suggests[1] that if $0 \leqslant \mu_i \leqslant c$ for all $i = 1, \ldots, s$ and some fixed $c$, then the MLBE $\hat{\mu}_0 \longrightarrow -\infty$ almost surely as $s \to \infty$. This in turn implies that the bias[2]

$$\hat{b}_0(\mu) \equiv E_\mu(\hat{\mu}_0) \to -\infty, \tag{6}$$

as $s \to \infty$. For this reason Hwang and Peddada (1994) stated that the MLE "fails disastrously", while Cohen and Sackrowitz (2002) deemed the MLE "undesirable" and proposed an alternative estimator $\tilde{\mu}_0$ (cf. Section 3) whose bias remains bounded.

Motivated by the classical Neyman and Scott (1948) example, however, Chaudhuri and Perlman (2003) viewed the tree-order estimation problem as a problem of estimating a single target parameter $\mu_0$ in the presence of an increasing number of "nuisance" parameters $\mu_1, \ldots, \mu_s$. As can be done for the Neyman–Scott example, Chaudhuri and Perlman proposed bias-reducing adjustments $\check{\mu}_0$ to the MLBE $\hat{\mu}_0$ that control the bias and improve upon (numerically) the Cohen–Sackrowitz estimator (CSE) in terms of mean-squared error (MSE).

However, this leaves open the question of the comparative behavior of the MLE and CSE for estimating the "nuisance" parameters $\mu_1, \ldots, \mu_s$. In this note we study this behavior and show that in most cases, as $s \to \infty$ the MLBEs $\hat{\mu}_1, \ldots, \hat{\mu}_s$ are asymptotically unbiased while the corresponding CSEs $\tilde{\mu}_1, \ldots, \tilde{\mu}_s$ are asymptotically biased. We conclude that for the version of

[1] Lee (1988) actually presents a slightly different result. However, see our Proposition 2.1.

[2] In fact, the bias $\hat{b}_0(\mu) \approx -\sqrt{2 \log s}$ as $s \to \infty$; cf. Chaudhuri and Perlman (2003), Eq. (24).