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Statistics & Probability Letters 71 (2005) 283–294
STATISTICS & PROBABILITY LETTERS
www.elsevier.com/locate/stapro

Reinforced weak convergence of stochastic processes

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Received 6 June 2002; received in revised form 1 November 2004
Available online 23 December 2004

Abstract

We consider a sequence of stochastic processes X_n on $C[0,1]$ converging weakly to X and call it *polynomially* convergent, if $\mathbf{E}F(X_n) \to \mathbf{E}F(X)$ for continuous functionals F of polynomial growth. We present a sufficient moment conditions on X_n for polynomial convergence and provide several examples, e.g. discrete excursions and depth first path associated to Galton–Watson trees. This concept leads to a new approach to moments of functionals of rooted trees such as height and path length.

Keywords: Weak convergence; Excursions; Functionals of trees

1. Introduction

Let X_n and X denote stochastic processes on $C[0,1]$. Then weak convergence of X_n to X means that for every continuous and bounded functional $F : C[0,1] \to \mathbf{R}$, we have

$$\mathbf{E}F(X_n) \to \mathbf{E}F(X), \quad (n \to \infty).$$

The topology on $C[0,1]$ is induced by the norm $\|.\|_\infty$. The purpose of this paper is to show that, under natural moments assumptions, this property can be extended to a wider class of functionals which need not be bounded anymore.

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doi:10.1016/j.spl.2004.11.005

For this purpose we introduce a notion of *reinforced* weak convergence.

Definition. Let X_n and X denote continuous stochastic processes on $C[0,1]$. We say that X_n converges polynomially to X if

$$\mathbf{E}F(X_n) \to \mathbf{E}F(X) \quad (n \to \infty) \tag{1}$$

for all continuous functionals $F : C[0,1] \to \mathbf{R}$ satisfying

$$|F(f)| \leqslant C(1 + \|f\|_\infty)^k \tag{2}$$

for some constants $C, k > 0$.

Remark 1. Examples for functionals F satisfying (2) are maximum and integrals. Hence, if X_n is polynomially convergent to X we have, for any $r > 0$ and for any Borel set $I \subset [0,1]$

$$\mathbf{E} \max_{t \in I} |X_n(t)|^r \to \mathbf{E} \max_{t \in I} |X(t)|^r$$

and

$$\mathbf{E} \int_I X_n(t)^r \, \mathrm{d}t \to \mathbf{E} \int_I X(t)^r \, \mathrm{d}t.$$

Remark 2. Note that polynomial convergence is equivalent to

$$\sup_n \mathbf{E}(\|X_n\|_\infty^k) < \infty \quad \text{(for all } k > 0) \tag{3}$$

if X_n converges weakly to X. This follows almost directly from Billingsley (1995, p. 338) (see also Lemmas 1 and 2).

We first state a necessary condition for polynomial convergence.

Theorem 1. *Suppose that X_n and X are stochastic processes on $C[0,1]$ such that X_n converges weakly to X. Further assume that the following two conditions are satisfied*:

1. *There exists $t_0 \in [0,1]$ such that for all integers $k \geqslant 0$*

$$\sup_n \mathbf{E}|X_n(t_0)|^k < \infty. \tag{4}$$

2. *There exists a sequence $(\alpha_d)_{d \geqslant 1}$ of positive real numbers with $\alpha_d > 1$ for infinitely many d such that for all integers $d > 0$*

$$\mathbf{E}|X_n(t) - X_n(s)|^d \leqslant C_d |t - s|^{\alpha_d} \quad \text{for all } s, t \in [0,1] \tag{5}$$

for some constant $C_d > 0$.

Then X_n converges polynomially to X.

Remark 3. Note that (4) and (5) imply tightness of X_n by applying Kolmogorov's criterion (see Revuz and Yor, 1999, p. 516). Furthermore, we will show in Section 2 that (4) and (5) imply $\sup_n \mathbf{E}|X_n(t_0)|^k < \infty$ for all fixed $t_0 \in [0,1]$ and integers $k \geqslant 0$. Consequently it follows from