Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Statistics & Probability Letters 75 (2005) 249–255
STATISTICS & PROBABILITY LETTERS
www.elsevier.com/locate/stapro

On convergence of random linear functionals

Suman Majumdar*

Department of Statistics, University of Connecticut, 1 University Place, Stamford, CT 06901-2315 USA

Received 20 April 2004; received in revised form 4 May 2005
Available online 29 August 2005

Abstract

When is the pointwise limit of the characteristic functions of a sequence $\{X_n\}$ of random elements taking values in a real separable Hilbert space H the characteristic function of a H valued random element? Uniform tightness of $\{\|X_n\| : n \geqslant 1\}$ is sufficient, but not necessary, whereas uniform tightness of $\{\langle X_n, y\rangle : n \geqslant 1, \|y\| \leqslant 1\}$ is necessary, but not sufficient. The sufficiency statement extends to a separable reflexive Banach space, whereas the necessity statement extends to a Banach space with separable dual.

MSC: primary 60B10; secondary 46C05; 46N30

Keywords: Skorohod–Dudley–Wichura representation; Prohorov compactness theorem; Uniform boundedness principle

1. Introduction and results

Let H be a real separable Hilbert space (isomorphic to l_2) with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\{x_n\}$ be a sequence in H such that

$$\langle x_n, y\rangle \to h(y) \quad \forall y \in H. \tag{1}$$

A necessary and sufficient condition for $h(y)$ to equal $\langle x, y\rangle$ for some $x \in H$ and for all $y \in H$ is

$$\{\|x_n\|\} \text{ is a bounded sequence in } \Re. \tag{2}$$

*Tel.: +1 203 251 8417; fax: +1 203 251 8592.
E-mail address: suman.majumdar@uconn.edu.

0167-7152/$ - see front matter
doi:10.1016/j.spl.2005.06.010

Since the linearity of $y \mapsto h(y)$ is immediate, the sufficiency of the condition follows from the Cauchy Schwartz inequality and the Riesz representation of H as its own dual, whereas the necessity of the condition is possibly the simplest application of the uniform boundedness principle. Our objective in this paper is to investigate the following question: if $\{x_n\}$ in (1) is replaced by a sequence of random elements $\{X_n\}$ and convergence in $\mathfrak{R}$ is replaced by convergence in law (of random variables, equivalently, probability distributions) in $\mathfrak{R}$, what is the *natural* replacement of (2)?

The results of our investigation, some of independent interest used to answer the question posed above, constitute the remainder of this section. Even though a necessary and sufficient condition eludes us, we do obtain a sufficient condition and a necessary condition, both of which reduce to (2) when the sequence consists of degenerate random elements.

Our first result (Lemma 1) puts together the vector space structure of measurable functions and the topology of convergence in probability.

Lemma 1. *Let $(\mathscr{X}, \mathscr{A}, P)$ be a probability space. Let F be a separable Frechet space. Let $\mathscr{L}_0(\mathscr{X}, F)$ denote the set of all F valued measurable (for $\mathscr{A}$ and the Borel σ-algebra of F) functions on $\mathscr{X}$. Let $L_0(\mathscr{X}, F)$ be the set of all equivalence classes of elements of $\mathscr{L}_0(\mathscr{X}, F)$ for the relation of P-*a.s. *equality. Let α denote the* Ky Fan *metric on $L_0(\mathscr{X}, F)$. Then $(L_0(\mathscr{X}, F), \alpha)$ is a Frechet space.*

Note that the completeness assertion of Lemma 1 is Theorem 9.2.3 of Dudley (1989), while the invariance of the *Ky Fan* metric is immediate from its definition (Dudley, 1989, Section 9.2) and the assumption that the range space F is Frechet.

Starting with convergent in law random linear functionals, equivalently, probability measures with linear functional induced distributions convergent in law, the next result (Proposition 1) constructs random elements with those probability distributions and convergent in probability linear functionals. In what follows, for any random element W, $\mathscr{L}(W)$ denotes the W induced distribution on the range space.

Proposition 1. *Let $\{X_n\}$ be a sequence of H valued random elements such that $\langle X_n, y\rangle$ converges in law to $\langle X, y\rangle$ for all $y \in H$, where X is another H valued random element. Then there exists a probability space $(\mathscr{X}, \mathscr{A}, P)$ and H valued random elements $\{Y_n\}$ and Y defined on $(\mathscr{X}, \mathscr{A}, P)$ such that $\mathscr{L}(Y_n) = \mathscr{L}(X_n)\ \forall n \geqslant 1$, $\mathscr{L}(Y) = \mathscr{L}(X)$, and $\langle Y_n, y\rangle$ converges to $\langle Y, y\rangle$ in probability for all $y \in H$.*

Next in line is the first of our two main results, obtainable by putting Lemma 1, Proposition 1 and the uniform boundedness principle together.

Theorem 1. *Let $\{X_n\}$ and X be as in Proposition 1. Then the family $\{\langle X_n, y\rangle : n \geqslant 1, \|y\| \leqslant 1\}$ of random variables is uniformly tight.*

Theorem 1 is our replacement for the necessity of (2). Theorem 2 will be our replacement for the sufficiency of (2).