Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Statistics & Probability Letters 72 (2005) 145–152
STATISTICS & PROBABILITY LETTERS
www.elsevier.com/locate/stapro

A general Ostrowski-type inequality ☆

J. de la Cal[a,*], J. Cárcamo[b]

[a]*Departamento de Matemática Aplicada y Estadística e Investigación Operativa, Facultad de Ciencia y Tecnología, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain*
[b]*Departamento de Economía Aplicada V, E.U. de Estudios Empresariales, Universidad del País Vasco, Elcano 21, 48008 Bilbao, Spain*

Received 27 February 2004; received in revised form 27 July 2004; accepted 20 December 2004

Abstract

We obtain an integral representation for the expectation which generalizes a well-known formula. As a consequence, we establish an estimate for the difference of two expectations which is optimal in a specific sense and is general enough to include as particular cases many of the Ostrowski-type inequalities existing in the literature. Other consequences concerning inequalities and stochastic orders are also discussed.

Keywords: Ostrowski inequality; Random variable; Distribution function; Expectation; Lebesgue integral; Absolutely continuous function; Convex function

1. Introduction and main results

As it is well known, if X is a real integrable random variable with distribution function F, then we have

$$EX = -\int_{-\infty}^{0} F(u)\,\mathrm{d}u + \int_{0}^{\infty} (1 - F(u))\,\mathrm{d}u.$$

☆ Research supported by the Spanish MCYT, Proyecto BFM2002-04163-C02-02, and by FEDER.
*Corresponding author. Tel.: +34 94 601 2654; fax: +34 94 601 3500.
E-mail addresses: mepcaagj@lg.ehu.es (J. de la Cal), mebcaurj@lg.ehu.es (J. Cárcamo).

0167-7152/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.spl.2004.12.013

Even, when X takes values in an interval I, we can write

$$EX = a - \int_{I_a^-} F(u)\,\mathrm{d}u + \int_{I_a^+} (1 - F(u))\,\mathrm{d}u, \tag{1}$$

where a is any fixed point of I, and

$$I_a^- := I \cap (-\infty, a], \quad I_a^+ := I \cap (a, \infty).$$

As a consequence, if Y is another I-valued integrable random variable with distribution function G, then we have

$$EY - EX = \int_I (F(u) - G(u))\,\mathrm{d}u. \tag{2}$$

In this paper, we generalize formulae (1) and (2) by giving integral representations for $Ef(X)$ and $Ef(Y) - Ef(X)$, when $f \in \mathscr{L}(I) :=$ the space of all real locally absolutely continuous functions on I. Thus, $f \in \mathscr{L}(I)$ means that the derivative f' exists almost everywhere, is locally integrable (i.e., integrable on each compact subinterval of I), and we can write, for all $x \in I$,

$$f(x) = f(a) - 1_{I_a^-}(x) \int_x^a f'(u)\,\mathrm{d}u + 1_{I_a^+}(x) \int_a^x f'(u)\,\mathrm{d}u, \tag{3}$$

where a is any fixed point of I, and 1_A stands for the indicator function of the set A. The aforementioned representations are expressed in terms of the principal value of an integral. We recall that, if g is a real locally integrable function on I, the principal value of the integral of g (on I) is defined by

$$⨍_I g(u)\,\mathrm{d}u := \lim_{n\to\infty} \int_{I_n} g(u)\,\mathrm{d}u, \tag{4}$$

where (I_n) is any increasing sequence of compact subintervals of I such that $\bigcup_n I_n = I$, provided that the (finite or infinite) limit exists and does not depend upon the particular sequence (I_n) involved. It is clear that

$$⨍_I g(u)\,\mathrm{d}u = \int_I g(u)\,\mathrm{d}u,$$

whenever g is integrable on I or g is nonnegative (nonpositive) a.e. on I.

Theorem 1. *Let X and Y be I-valued random variables with distribution functions F and G, respectively, and let $f \in \mathscr{L}(I)$ be such that $f(X)$ and $f(Y)$ are integrable. Then, we have*

$$Ef(X) = f(a) - ⨍_{I_a^-} f'(u)F(u)\,\mathrm{d}u + ⨍_{I_a^+} f'(u)(1 - F(u))\,\mathrm{d}u \tag{5}$$

(*where a is any fixed point of I*), *and*

$$Ef(Y) - Ef(X) = ⨍_I f'(u)(F(u) - G(u))\,\mathrm{d}u. \tag{6}$$

Remark 1. In the setting of the preceding theorem, the principal value cannot be, in general, replaced by a proper integral. Take, for instance, $f(x) := -x \cos x + \sin x$, and let X be a random