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Bootstrapping modified goodness-of-fit statistics with estimated parameters

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Abstract

Goodness-of-fit tests are proposed for testing a composite null hypothesis that is a general parametric family of distribution functions. They are distribution-free under the null hypothesis and have a limiting normal distribution under the null and the alternative hypothesis. To avoid the estimation of the asymptotic variance under the alternative hypothesis, we propose consistent bootstrap estimators.

MSC: primary 62G09; 62G10

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1. Introduction

In this paper we consider, for a random sample from F, the goodness-of-fit hypothesis $H_0: F = G$, where G is a given general parametric family of distribution functions, containing unknown parameters $\boldsymbol{\theta}$ (e.g. location, scale, etc.) which have to be estimated. It is well known that in the situation where $\boldsymbol{\theta}$ is known, the classical test statistics typically take the form of a degenerate U- or V-statistic and that the limiting null distribution is that of a (possibly infinite) sum of

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weighted chi-squared variables. Finding the weights is not easy since they are the eigenvalues of some operator equation and they can only be found in some special cases. We avoid this problem by using some slight modification of the empirical distribution function in the construction of our test statistic. A similar idea has been used in Ahmad (1993, 1996) and Ahmad and Alwasel (1999). This leads to test statistics that have a limiting normal distribution with the usual $n^{1/2}$ standardization both under the null and the alternative hypothesis (Section 3). The first main objective of our paper is to provide conditions under which the replacement of the unknown $\boldsymbol{\theta}$ by a suitable estimator $\widehat{\boldsymbol{\theta}}$ keeps asymptotic normality in force (Section 4). It turns out that the statistic with estimated nuisance parameter has the same limit distribution under H_0. The question of replacing the unknown $\boldsymbol{\theta}$ by an estimator has not been dealt with in the above references. The problem has been considered by De Wet and Randles (1987) in the unmodified case and our result provides an alternative to their paper. Our approach has the advantage of also providing the limit behavior under the alternative hypothesis H_1: $F \neq G$. The second main objective of our paper is to establish the validity of bootstrap approximations; this provides a way to avoid the estimation of the complicated and unknown variance parameter in the asymptotic distribution under the alternative hypothesis H_1 (Sections 5 and 6). See Koul and Lahiri (1994) for a similar approach in a regression context. Our proposed resampling scheme is nonparametric and works under H_0 and H_1. This is more general than the parametric bootstrap in a recent paper of Jiménez-Gamero et al. (2003), who only prove consistency under H_0. We begin, in Section 2, with a useful characterization for the equality of two continuous distribution functions.

2. Characterization

Assume that F and G are continuous distribution functions. The problem of testing the hypothesis H_0: $F = G$ versus H_1: $F \neq G$ is often based on the L_2-distance $\int (F - G)^2 \, \mathrm{d}G$ or more generally on the L_{2p}-distance $\int (F - G)^{2p} \, \mathrm{d}G$ for some integer $p \geqslant 1$. If $X_1, \ldots, X_n$ is a random sample from F, then an obvious test statistic is given by $\int (F_n - G)^{2p} \, \mathrm{d}G$, where F_n is the usual empirical distribution function of the sample. The case $p = 1$ is the well-known Cramér-von Mises statistic. Using the binomial expansion and some integration by parts, it is easily shown that the L_{2p}-distance statistics can be rewritten and that this leads to the following alternative characterizations of the null hypothesis: we have that $F = G$ if and only if

$$\sum_{i=2}^{2p} \binom{2p}{i} \frac{(-1)^{2p-i}}{2p-i+1} E[G^{2p-i+1}(\max(X_1, \ldots, X_i))] - E[G^{2p}(X_1)] = \frac{1}{2p+1}$$

(for some $p = 1, 2, \ldots$).

In the present paper we will only work with the $p = 1$ version of this characterization, which takes the simple form

$$E[G(\max(X_1, X_2))] - E[G^2(X_1)] = \tfrac{1}{3}. \tag{1}$$

Note that characterization (1) also implicitly appears in Too and Lin (1989) via a totally different approach.