Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 126 (2016) 315–336

stochastic processes and their applications

www.elsevier.com/locate/spa

Asymptotic proportion of arbitrage points in fractional binary markets

Fernando Cordero[a,*], Irene Klein[b], Lavinia Perez-Ostafe[c]

[a] *Faculty of Technology, University of Bielefeld, Universitätsstr. 25, 33615 Bielefeld, Germany*
[b] *Department of Statistics and Operations Research, University of Vienna, Oskar-Morgensternplatz 1, A-1090 Vienna, Austria*
[c] *Department of Mathematics, ETH Zurich, Rämistrasse 101, 8092 Zurich, Switzerland*

Received 13 August 2014; received in revised form 30 March 2015; accepted 1 September 2015
Available online 10 September 2015

Abstract

A fractional binary market is a binary model approximation for the fractional Black–Scholes model, which Sottinen constructed with the help of a Donsker-type theorem. In a binary market the non-arbitrage condition is expressed as a family of conditions on the nodes of a binary tree. We call "arbitrage points" the nodes which do not satisfy such a condition and "arbitrage paths" the paths which cross at least one arbitrage point. In this work, we provide an in-depth analysis of the asymptotic proportion of arbitrage points and arbitrage paths. Our results are obtained by studying an appropriate rescaled disturbed random walk.

Keywords: Fractional Brownian motion; Fractional binary markets; Binary markets; Arbitrage opportunities

1. Introduction

In the classical theory of mathematical finance a crucial role is played by the notion of *arbitrage*, which is the cornerstone of the option pricing theory that goes back to F. Black, R. Merton and M. Scholes [2]. In the case of binary markets, the absence of arbitrage is completely characterized by Dzhaparidze in [6]. Intuitively, a binary market is a market in which the stock price

* Corresponding author.
E-mail addresses: fcordero@techfak.uni-bielefeld.de (F. Cordero), irene.klein@univie.ac.at (I. Klein), lavinia.perez@math.ethz.ch (L. Perez-Ostafe).

http://dx.doi.org/10.1016/j.spa.2015.09.002
0304-4149/

process $(S_n)_{n=0}^N$ is an adapted stochastic process with strictly positive values and such that at time n the stock price evolves from S_{n-1} to either $\alpha_n S_{n-1}$ or $\beta_n S_{n-1}$, where $\beta_n < \alpha_n$.

One advantage of working with binary markets is given, on one hand, by their simplicity and, on the other hand, by their flexibility to approximate more complicated models. In particular, this is possible for Black–Scholes type markets that are driven by a process, for which we dispose of a random walk approximation. Examples of this are the fractional Brownian motion and the Rosenblatt process, as one can see in [14,15] respectively.

In this paper we provide an in-depth analysis of fractional binary markets, which are defined by Sottinen [14] as a sequence of binary models approximating the fractional Black–Scholes model, i.e. a Black–Scholes type model where the randomness of the risky asset comes from a fractional Brownian motion. Along this work we assume that the Hurst parameter H is strictly bigger than $1/2$. In this case, the fractional Brownian motion exhibits self-similarity and long-range dependence, properties that were observed in some empirical studies of financial time series (see [3,17]). Since the fractional Brownian motion fails to be a semimartingale, the fractional Black–Scholes model admits arbitrage opportunities, a drawback that can be corrected if, e.g. one introduces transaction costs.

In [14] Sottinen constructs the fractional binary markets by giving an analogue of the Donsker theorem, where the fractional Brownian motion is approximated in distribution by a "disturbed" random walk. Sottinen proves that the arbitrage opportunities do not only appear in the limiting model, but also in the sequence of fractional binary markets.

According to [6], in a binary market, the absence of arbitrage can be written as a family of conditions on the nodes of a binary tree. We call an "arbitrage point" a node in the binary tree which does not satisfy the corresponding non-arbitrage condition. An "arbitrage path" is a path that crosses at least one arbitrage point. By [14] we know that, for each fractional binary market in the sequence, the associated set of arbitrage points is not empty.

The study of the set of arbitrage points provides a way to quantify arbitrage, a research direction which goes a step further than the classical question of existence of arbitrage.

The aim of this paper is to study qualitative and quantitative properties of the sets of arbitrage points and paths for the fractional binary market. First, we prove that starting from any point in the binary tree we reach an arbitrage point by going enough times only up or only down (Proposition 3.2). This generalizes the result of Sottinen, who showed the existence of arbitrage starting only from the root of the tree. This gives information about the structure of the set of arbitrage points and implies that its cardinality is asymptotically infinite. Next, we study the limit behaviour of the proportion of arbitrage points. The latter is expressed in terms of a rescaled random walk, which we show converges in law. The characterization of the asymptotic proportion of arbitrage points then follows (Theorem 3.3). We also take a closer look to the previous limit when H tends to $1/2$ and when H tends to 1 (Proposition 3.4). Finally, making use of the 0–1 Kolmogorov law, we show that when H is close to 1, a.s. a path in the binary tree crosses an infinite number of arbitrage points, and when H is close to $1/2$, a.s. a path in the binary tree crosses an infinite number of non-arbitrage points (Theorem 3.5).

We believe that our asymptotic results open a way to a better understanding of the arbitrage behaviour in the limiting market. Since the proportion of arbitrage points remains strictly positive in the limit, one could expect that the sequence of sets of arbitrage points converges in a proper way to a set encoding the arbitrage structure of the fractional Black–Scholes model.

Another possible direction, in which our results may turn useful is the study of arbitrage in the fractional binary markets under transaction costs. As mentioned by Sottinen, one may expect that the arbitrage disappears when transaction costs are taken into account. This latter problem