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Iterated random functions and slowly varying tails

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Abstract

Consider a sequence of i.i.d. random Lipschitz functions $\{\Psi_n\}_{n\geq 0}$. Using this sequence we can define a Markov chain via the recursive formula $R_{n+1} = \Psi_{n+1}(R_n)$. It is a well known fact that under some mild moment assumptions this Markov chain has a unique stationary distribution. We are interested in the tail behaviour of this distribution in the case when $\Psi_0(t) \approx A_0 t + B_0$. We will show that under subexponential assumptions on the random variable $\log^+(A_0 \vee B_0)$ the tail asymptotic in question can be described using the integrated tail function of $\log^+(A_0 \vee B_0)$. In particular we will obtain new results for the random difference equation $R_{n+1} = A_{n+1}R_n + B_{n+1}$.

MSC: 60H25; 60J10

Keywords: Stochastic recursions; Random difference equation; Stationary distribution; Subexponential distributions

1. Introduction

Consider a sequence of independent identically distributed (i.i.d.) random Lipschitz functions $\{\Psi_n\}_{n\geq 0}$, where $\Psi_n : \mathbb{R} \to \mathbb{R}$ for $n \in \mathbb{N}$. Using this sequence we can define a Markov chain via the recursive formula

$$R_{n+1} = \Psi_{n+1}(R_n) \quad \text{for } n \geq 0, \tag{1.1}$$

where $R_0 \in \mathbb{R}$ is arbitrary but independent of the sequence $\{\Psi_n\}_{n\geq 0}$. Put $\Psi = \Psi_0$. We are interested in the existence and properties of the stationary distribution of the Markov chain

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$\{R_n\}_{n\geq 0}$, that is the solution of the stochastic fixed point equation

$$R \stackrel{d}{=} \Psi(R) \quad R \text{ independent of } \Psi, \tag{1.2}$$

where the distribution of random variable R is the stationary distribution of the Markov chain $\{R_n\}_{n\geq 0}$.

The main example, we have in mind, is the random difference equation, where Ψ is an affine transformation, that is $\Psi_n(t) = A_n t + B_n$ with $\{(A_n, B_n)\}_{n\geq 0}$ being an i.i.d. sequence of two-dimensional random vectors. Then the formula (1.1) can be written as

$$R_{n+1} = A_{n+1} R_n + B_{n+1} \quad \text{for } n \geq 0. \tag{1.3}$$

Put $(A, B) = (A_0, B_0)$. It is a well known fact that if

$$\mathbb{E}[\log |A|] < 0 \quad \text{and} \quad \mathbb{E}[\log^+ |B|] < \infty,$$

then the Markov chain $\{R_n\}_{n\geq 0}$ given by (1.3) has a unique stationary distribution which can be represented as the distribution of the random variable

$$R = \sum_{n\geq 0} B_{n+1} \prod_{k=1}^{n} A_k, \tag{1.4}$$

for details see [28]. Random variables of this form can be found in analysis of probabilistic algorithms or financial mathematics, where R would be called a perpetuity. Such random variables occur also in number theory, combinatorics, as a solution to stochastic fixed point equation

$$R \stackrel{d}{=} AR + B \quad R \text{ independent of } (A, B), \tag{1.5}$$

atomic cascades, random environment branching processes, exponential functionals of Lévy processes, Additive Increase Multiplicative Decrease algorithms [17], COGARCH processes [22], and more. A variety of examples for possible applications of R can be found in [14,15,11].

From the application point of view, the key information is the behaviour of the tail of R, that is

$$\mathbb{P}[R > x] \quad \text{as } x \to \infty.$$

This problem was investigated by various authors, for example by Goldie and Grübel [14] and in a similar setting by Hitczenko and Wesołowski [18]. The first result says that if B is bounded, $\mathbb{P}[A \in [0, 1]] = 1$ and the distribution of A behaves like the uniform distribution in the neighbourhood of 1, then R given by (1.4) has thin tail, more precisely $\log \mathbb{P}[R \geq x] \sim -cx \log(x)$. Recall that for two positive functions $f(\cdot)$ and $g(\cdot)$, by $f(x) \sim g(x)$ we mean that $\lim_{x\to\infty} f(x)/g(x) = 1$. In this paper we are only interested in limits as $x \to \infty$, so from now we omit the specification of the limit.

There is also the result of Kesten [20] and later on, in the same setting, of Goldie [13]. The essence of this result is that under Cramér's condition, that is if $\mathbb{E}[|A|^\alpha] = 1$ for some $\alpha > 0$ such that $\mathbb{E}[|B|^\alpha] < \infty$, the tail of R is regularly varying, i.e. $\mathbb{P}[R > x] \sim cx^{-\alpha}$ for some positive and finite constant c and R defined by (1.4).

Finally, the result of Grincevičius [16], which was later generalised by Grey [15], states that in the case of positive A if for some $\alpha > 0$ we have $\mathbb{E}[A^\alpha] < 1$ and $\mathbb{P}[B > x] \sim x^{-\alpha} L(x)$, where L is slowly varying (that is $L(cx) \sim L(x)$ for any positive c), then the tail of R is again regularly varying, in fact $\mathbb{P}[R > x] \sim cx^{-\alpha} L(x)$. Note that in this case the tail of perpetuity R exhibits the same rate of decay as the tail of the input, that is $\mathbb{P}[R > x] \sim c\mathbb{P}[B > x]$.