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Random mass splitting and a quenched invariance principle

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Abstract

We will investigate a random mass splitting model and the closely related random walk in a random environment (RWRE). The heat kernel for the RWRE at time t is the mass splitting distribution at t. We prove a quenched invariance principle (QIP) for the RWRE which gives us a quenched central limit theorem for the mass splitting model. Our RWRE has an environment which is changing with time. We follow the outline for proving a QIP for a random walk in a space–time random environment laid out by Rassoul-Agha and Seppäläinen (2005) which in turn was based on the work of Kipnis and Varadhan (1986) and others.

Keywords: Random walk in random environment; Invariance principle

1. Introduction

Imagine a one dimensional city (motivated by Abbott's *Flatland*) on the Y-axis with houses at the points $(0, k)$. Suppose the city is stricken with an epidemic and things are getting worse by the day. As the death toll rises, each house $(0, k)$ has only a Poisson(1) number of survivors $v(0, k)$ when the long awaited medical breakthrough suddenly happens. The surviving residents (assume at least one survives) of house $(0, 0)$ are scientists who were working on a cure for a while and they finally have an antidote! But the quantity that is produced is limited (say has mass 1 unit). So, they get out of their house carrying an equal proportion of the medicine $(1/v(0, 0))$, and

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start performing one-dimensional simple random walks, with time represented along the positive X-axis and the spatial coordinate along the Y-axis, to share it with the survivors. Meanwhile, the remaining survivors decide that staying inside the house in unhygienic conditions is dangerous, and they get out of their respective houses and start performing (one-dimensional) simple random walks at the same time as the scientists. Whenever a group of people meet on the way, say at (t, k), and at least one person in the group has some medicine, it gets equally divided among all the people in the group. This process continues. The question we ask in this article is: How does the mass distribution of medicine at time t behave for large values of t?

We call this system the *Random Mass Splitting* model. This model came about as the natural discrete analogue of the following problem which was suggested to us by Krzysztof Burdzy [8]: suppose there is an initial configuration of particles distributed according to a Poisson point process. Each particle performs a Brownian motion when away from its two neighboring particles, and when two particles meet, they get reflected off each other so that the *ordering is preserved*. This is the two-sided version of the *Atlas model* (see [23]) which frequently appears in Stochastic Portfolio theory. We give a mass of one to a tagged particle. Every time two particles collide, the mass splits in half between them (it can be shown that there are no triple collisions by methods of [23]). A way of understanding the intersection graph of these particles is to monitor the evolution of this mass in time. We hope that the developments of this paper will shed light on this problem.

Our model fits in a well-studied class of models where two types of particles interact. These problems become substantially harder when *both types of particles move*. In [17–19] Kesten and Sidoravicius investigated the shape of the infected set when a group of moving particles spread a contagious disease (or rumor) among another group of healthy (ignorant) particles which also move. The movement of each particle is assumed to be independent of the other. Another example from this class is the work of Peres et al. [25] who studied the detection of a moving particle by a mobile wireless network. Our model is more in the spirit of the former collection of papers. However instead of looking at a shape theorem for the sites where the antidote has reached, we are interested in *quantifying the spread* by looking at the distribution of mass that is initially carried by a few particles (the ones at the origin).

Our study of the model begins with the following observation. We use the movement of the villagers over time to define a random environment. Then we study the movement of a random walker in this random environment (RWRE). We study this RWRE because it turns out that the heat kernel of the walk conditioned on the environment ω at time t is precisely the distribution of the medicine at time t when the villagers are moving according to ω. Random walk in random environment models have been studied by many authors. This can be a very difficult field as even the simplest properties such as transience and recurrence are difficult to establish [16]. But a theory of random walk in random environments has been developed. We will harness that theory to find the asymptotic mass distribution.

In many examples, such as simple random walk on supercritical percolation clusters, we get a quenched invariance principle (i.e. an almost sure convergence of the RWRE to Brownian motion) [1,30]. However in other examples we find behaviors that are very different from the usual diffusive behavior of simple random walk on $\mathbb{Z}^d$ [31,2]. Many of the proofs of invariance principles for random walks in random environments have their origins in the seminal work of Kipnis and Varadhan [20]. This paper laid down the foundation for quenched invariance principles for reversible Markov chains. Maxwell and Woodroofe [21] and Derriennic and Lin [9] subsequently extended their approach to the non-reversible set-up. Rassoul-Agha and Seppäläinen [27] developed further on these techniques to give a set of conditions under