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On the $\frac{1}{H}$-variation of the divergence integral with respect to fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$

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Abstract

In this paper, we study the $\frac{1}{H}$-variation of stochastic divergence integrals $X_t = \int_0^t u_s \delta B_s$ with respect to a fractional Brownian motion B with Hurst parameter $H < \frac{1}{2}$. Under suitable assumptions on the process u, we prove that the $\frac{1}{H}$-variation of X exists in $L^1(\Omega)$ and is equal to $e_H \int_0^T |u_s|^{\frac{1}{H}} ds$, where $e_H = \mathbb{E}\left[|B_1|^{\frac{1}{H}}\right]$. In the second part of the paper, we establish an integral representation for the fractional Bessel Process $\|B_t\|$, where B_t is a d-dimensional fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$. Using a multidimensional version of the result on the $\frac{1}{H}$-variation of divergence integrals, we prove that if $2dH^2 > 1$, then the divergence integral in the integral representation of the fractional Bessel process has a $\frac{1}{H}$-variation equals to a multiple of the Lebesgue measure.

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1. Introduction

The fractional Brownian motion (fBm for short) $B = \{B_t, t \in [0, T]\}$ with Hurst parameter $H \in (0, 1)$ is a Gaussian self-similar process with stationary increments. This process was introduced by Kolmogorov [13] and studied by Mandelbrot and Van Ness in [16], where a stochastic integral representation in terms of a standard Brownian motion was established. The parameter H is called Hurst index from the statistical analysis, developed by the climatologist Hurst [11]. The self-similarity and stationary increments properties make the fBm an appropriate model for many applications in diverse fields from biology to finance. From the properties of the fBm, it follows that for every $\alpha > 0$

$$\mathbb{E}\left(|B_t - B_s|^{\alpha}\right) = \mathbb{E}\left(|B_1|^{\alpha}\right) |t - s|^{\alpha H}.$$

As a consequence of the Kolmogorov continuity theorem, we deduce that there exists a version of the fBm B which is a continuous process and whose paths are γ-Hölder continuous for every $\gamma < H$. Therefore, the fBm with Hurst parameter $H \neq \frac{1}{2}$ is not a semimartingale and then the Itô approach to the construction of stochastic integrals with respect to fBm is not valid. Two main approaches have been used in the literature to define stochastic integrals with respect to fBm with Hurst parameter H. Pathwise Riemann–Stieltjes stochastic integrals can be defined using Young's integral [20] in the case $H > \frac{1}{2}$. When $H \in (\frac{1}{4}, \frac{1}{2})$, the rough path analysis introduced by Lyons [15] is a suitable method to construct pathwise stochastic integrals.

A second approach to develop a stochastic calculus with respect to the fBm is based on the techniques of Malliavin calculus. The divergence operator, which is the adjoint of the derivative operator, can be regarded as a stochastic integral, which coincides with the limit of Riemann sums constructed using the Wick product. This idea has been developed by Decreusefond and Üstünel [6], Carmona, Coutin and Montseny [4], Alòs, Mazet and Nualart [1,2], Alòs and Nualart [3] and Hu [8], among others. The integral constructed by this method has zero mean. Different versions of Itô's formula have been proved by the divergence integral in these papers. In particular, if $H \in (\frac{1}{4}, 1)$ and $f \in C^2(\mathbb{R})$ is a real-valued function satisfying some suitable growth condition, then the stochastic process $\{f'(B_t)\mathbf{1}_{[0,t]}, 0 \le t \le T\}$ belongs to domain of the divergence operator and

$$f(B_t) = f(0) + \int_0^t f'(B_s)\delta B_s + H \int_0^t f''(B_s)s^{2H-1}ds. \tag{1}$$

For $H \in (0, \frac{1}{4}]$, this formula still holds, if the stochastic integral is interpreted as an extended divergence operator (see [5,14]). A multidimensional version of the change of variable formula for the divergence integral has been recently proved by Hu, Jolis and Tindel in [9].

Using the self-similarity of fBm and the Ergodic Theorem one can prove that the fBm has a finite $\frac{1}{H}$-variation on any interval $[0, t]$, equals to $e_H t$, where $e_H = \mathbb{E}\left[|B_1|^{\frac{1}{H}}\right]$ (see, for instance, Rogers [18]). More precisely, we have, as n tends to infinity

$$\sum_{i=0}^{n-1} |B_{t(i+1)/n} - B_{it/n}|^{\frac{1}{H}} \xrightarrow{L^1(\Omega)} t\, e_H. \tag{2}$$

This result has been generalized by Guerra and Nualart [7] to the case of divergence integrals with respect to the fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$.