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Convex hulls of random walks and their scaling limits

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Abstract

For the perimeter length and the area of the convex hull of the first n steps of a planar random walk, we study $n \to \infty$ mean and variance asymptotics and establish non-Gaussian distributional limits. Our results apply to random walks with drift (for the area) and walks with no drift (for both area and perimeter length) under mild moments assumptions on the increments. These results complement and contrast with previous work which showed that the perimeter length in the case with drift satisfies a central limit theorem. We deduce these results from weak convergence statements for the convex hulls of random walks to scaling limits defined in terms of convex hulls of certain Brownian motions. We give bounds that confirm that the limiting variances in our results are non-zero.

MSC: primary 60G50; 60D05; secondary 60J65; 60F05; 60F17

Keywords: Convex hull; Random walk; Brownian motion; Variance asymptotics; Scaling limits

1. Introduction

Random walks are classical objects in probability theory. Recent attention has focused on various geometrical aspects of random walk trajectories. Many of the questions of stochastic geometry, traditionally concerned with functionals of independent random points, are also of interest for point sets generated by random walks. Here we examine the asymptotic behaviour of

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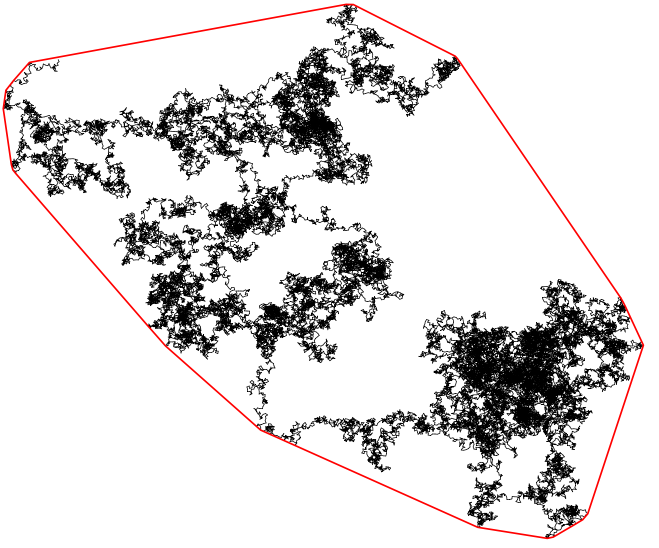

Fig. 1. Simulated path of a zero-drift random walk and its convex hull.

the *convex hull* of the first n steps of a random walk in $\mathbb{R}^2$, a natural geometrical characteristic of the process. Study of the convex hull of planar random walk goes back to Spitzer and Widom [21] and the continuum analogue, convex hull of planar Brownian motion, to Lévy [15, §52.6, pp. 254–256]; both have received renewed interest recently, in part motivated by applications arising for example in modelling the 'home range' of animals. See [16] for a recent survey of motivation and previous work. The method of the present paper in part relies on an analysis of *scaling limits*, and thus links the discrete and continuum settings.

Let Z be a random vector in $\mathbb{R}^2$, and let $Z_1, Z_2, \ldots$ be independent copies of Z. Set $S_0 := 0$ and $S_n := \sum_{k=1}^n Z_k$; S_n is the planar random walk, started at the origin, with increments distributed as Z. We will impose a moments condition of the following form:

(M_p) Suppose that $\mathbb{E}[\|Z\|^p] < \infty$.

Throughout the paper we assume (usually tacitly) that the $p = 2$ case of (M_p) holds. For several of our results we impose a stronger condition and assume that (M_p) holds for some $p > 2$, in which case we say so explicitly.

Given (M_p) holds for some $p \geq 2$, both $\mu := \mathbb{E}Z \in \mathbb{R}^2$, the mean drift vector of the walk, and $\Sigma := \mathbb{E}[(Z-\mu)(Z-\mu)^\top]$, the covariance matrix associated with Z, are well defined; Σ is positive semidefinite and symmetric. We also write $\sigma^2 := \operatorname{tr} \Sigma = \mathbb{E}[\|Z-\mu\|^2]$. Here and elsewhere Z and μ are viewed as column vectors, and $\|\cdot\|$ is the Euclidean norm.

For a subset $\mathcal{S}$ of $\mathbb{R}^d$, its convex hull, which we denote hull $\mathcal{S}$, is the smallest convex set that contains $\mathcal{S}$. We are interested in hull $\{S_0, S_1, \ldots, S_n\}$, which is a (random) convex polygon, and in particular in its perimeter length L_n and area A_n. (See Fig. 1.)

The perimeter length L_n has received some attention in the literature, initiated by the remarkable formula of Spitzer and Widom [21], which states that

$$\mathbb{E}L_n = 2\sum_{k=1}^{n} k^{-1}\mathbb{E}\|S_k\|, \quad \text{for all } n \in \mathbb{N} := \{1, 2, \ldots\}. \tag{1}$$

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