Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 124 (2014) 3312–3341

stochastic processes and their applications

www.elsevier.com/locate/spa

Phase reduction in the noise induced escape problem for systems close to reversibility

Christophe Poquet*

Université Paris Diderot (Paris 7) and Laboratoire de Probabilités et Modèles Aléatoires (CNRS), U.F.R. Mathématiques, Case 7012, 75205 Paris Cedex 13, France

Received 15 July 2013; received in revised form 8 May 2014; accepted 8 May 2014
Available online 16 May 2014

Abstract

We consider n-dimensional deterministic flows obtained by perturbing a gradient flow. We assume that the gradient flow admits a stable curve of stationary points, and thus if the perturbation is not too large the perturbed flow also admits an attracting curve. We show that the noise induced escape problem from a stable fixed point of this curve can be reduced to a one-dimensional problem: we can approximate the associated quasipotential by the one associated to the restricted dynamics on the stable curve. The error of this approximation is given in terms of the size of the perturbation.

MSC: 60F10; 37N25

Keywords: Phase reduction; Escape problem; Large deviations; Quasipotential; Optimal path; Normally hyperbolic manifolds

1. Introduction

1.1. Phase reduction and escape problem

For dynamical systems with an attracting limit cycle, the phase reduction method consists in simplifying the system by projecting the dynamics on the limit cycle, and neglecting the distance

* Correspondence to: Université Paris Dauphine and CEREMADE, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France. Tel.: +33 6 73 64 70 06.

E-mail address: poquet@math.univ-paris-diderot.fr.

http://dx.doi.org/10.1016/j.spa.2014.05.003
0304-4149/

between the trajectory and the limit cycle [13]. Such an approximation allows to reduce the dynamics to a one dimensional self-contained equation satisfied by the phase parametrizing the limit cycle. Such a reduction is widely used in the context of noisy oscillators (see [9,18,19] and references therein).

The aim of this paper is to show that the phase reduction can be made in a rigorous way for the escape problem for a class of systems close to reversibility. For a smooth dynamical system

$$\mathrm{d}X_t = F[X_t]\,\mathrm{d}t, \tag{1.1}$$

where $X_t \in \mathbb{R}^n$ (we use the notation $f[\cdot]$ for functions with domain $\mathbb{R}^n$), including a stable fixed point A with basin of attraction D, the escape problem is the study of the metastable behavior of A under a small noisy perturbation

$$\mathrm{d}X_t = F[X_t]\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t, \tag{1.2}$$

where B_t is a Brownian motion in $\mathbb{R}^n$. The natural questions arising are where, when and how do trajectories of (1.2) escape from D. We aim to show that for a class of systems close to reversibility and containing an attracting curve M, the escape from a stable fixed point A located on M occurs close to M, and that the probability of this escape can be well approximated by considering only the dynamics constrained to M.

The problem of noisy escape from a fixed point has been much studied in the literature. The fundamental reference is of course [7], where it is shown that these questions are related to the large deviation behavior of (1.2), and more precisely to the corresponding "quasipotential". For a connected domain K of $\mathbb{R}^n$ and two points P_1 and P_2 of K, the quasipotential $W_K(P_1, P_2)$ is defined by

$$W_K(P_1, P_2) = \inf\left\{ I_T^{P_1}(Y) : \, Y \in C([T, 0], K), \; T < 0, \; Y_T = P_1, \; Y_0 = P_2 \right\}, \tag{1.3}$$

where I is the large deviation rate function of (1.2), that is

$$I_T^x(Y) = \begin{cases} \dfrac{1}{2}\displaystyle\int_T^0 \left\| \dot{Y}_t - F[Y_t] \right\|^2 \mathrm{d}t & \text{if } Y \text{ is absolutely continuous} \\ & \text{and } Y_T = x, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.4}$$

where $\|\cdot\|$ is the norm associated to the canonical scalar product $\langle\cdot,\cdot\rangle$ of $\mathbb{R}^n$. If K is a compact neighborhood of A with smooth boundaries included in D the escape from K will take place, with probability tending to 1 as $\varepsilon \to 0$, very close to the points B of the boundary of K satisfying $W_K(A, B) = \inf_{E \in \partial K} W_K(A, E)$ [7,3,15,16]. By a compactness argument and since $W_K(A, \cdot)$ is continuous [7], there exists at least one point B satisfying this property. Moreover for each starting point $x \in K$, the exit time τ^ε satisfies

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_x \tau^\varepsilon = W_K(A, B), \tag{1.5}$$

and this escape time is in fact, after renormalization, asymptotically exponential (e.g. [3,15,16]). So in the case we are interested in, to show that the escape trajectories stay close to the attracting curve M, it is sufficient to prove that for a thin tubular neighborhood U of M the minimum of the quasipotential on ∂U is achieved at one the two tips of U. We will also show that this minimum of quasipotential on ∂U can be well approximated by the quasipotential corresponding to the dynamics constrained to M.