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Some limit theorems for Hawkes processes and application to financial statistics

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Abstract

In the context of statistics for random processes, we prove a law of large numbers and a functional central limit theorem for multivariate Hawkes processes observed over a time interval $[0, T]$ when $T \to \infty$. We further exhibit the asymptotic behaviour of the covariation of the increments of the components of a multivariate Hawkes process, when the observations are imposed by a discrete scheme with mesh Δ over $[0, T]$ up to some further time shift τ. The behaviour of this functional depends on the relative size of Δ and τ with respect to T and enables to give a full account of the second-order structure. As an application, we develop our results in the context of financial statistics. We introduced in Bacry et al. (2013) [7] a microscopic stochastic model for the variations of a multivariate financial asset, based on Hawkes processes and that is confined to live on a tick grid. We derive and characterise the exact macroscopic diffusion limit of this model and show in particular its ability to reproduce the important empirical stylised fact such as the Epps effect and the lead–lag effect. Moreover, our approach enables to track these effects across scales in rigorous mathematical terms.

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1. Introduction

1.1. Motivation and setting

Point processes have long served as a representative model for event-time based stochastic phenomena that evolve in continuous time. A comprehensive mathematical development of the theory of point processes can be found in the celebrated textbook of Daley and Vere-Jones [12]; see also the references therein. In this context, mutually exciting processes form a specific but quite important class of point processes that are mathematically tractable and widely used in practice. They were first described by Hawkes in 1971 [17,16]. Informally, to a multivariate d-dimensional counting process $N = (N_1, \dots, N_d)$ with values in $\mathbb{N}^d$ is associated an intensity function $(\lambda_1, \dots, \lambda_d)$ defined by

$$P\big(N_i \text{ has a jump in } [t, t+dt] \mid \mathcal{F}_t\big) = \lambda_{i,t}\, dt, \quad i = 1, \dots, d$$

where P stands for probability and $\mathcal{F}_t$ is the sigma-field generated by N up to present time t. A multivariate Hawkes process has intensity

$$\lambda_{i,t} = \mu_i + \int_{(0,t)} \sum_{j=1}^{d} \varphi_{ij}(t-s) dN_{j,s}, \quad i = 1, \dots, d \tag{1}$$

and is specified by $\mu_i \in \mathbb{R}_+ = [0, \infty)$ and for $i = 1, \dots, d$, the φ_{ij} are functions from $\mathbb{R}_+$ to $\mathbb{R}_+$. More in Section 2 for rigorous definitions.

The properties of Hawkes processes are fairly well known: from a probabilistic point of view, the aforementioned book of Daley and Vere-Jones [12] gives a concise synthesis of earlier results published by Hawkes [17,16,18] that focus on spectral analysis following Bartlett [9] and cluster representation; see Hawkes and Oakes [19]. From a statistical perspective, Ogata studied in [24] the maximum likelihood estimator. Recently, the nonparametric estimation of the intensity functions has been investigated by Reynaud-Bouret and Schbath [25] and Al Dayri et al. [13].

However, in all these papers and the references therein, the focus is on the "microscopic properties" of Hawkes processes, *i.e.* their infinitesimal evolution, possibly under a stationary regime or for a large time horizon $[0, T]$ in order to guarantee a large number of jumps for statistical inference purposes. In the present work, we are rather interested in the "macroscopic properties" of Hawkes processes, in the sense of obtaining a limit behaviour for the multivariate process $(N_{Tv})_{v\in[0,1]}$ as $T \to \infty$, for a suitable normalisation. Our interest originates in financial data modelling: in [7], we introduced a stochastic model for the price $S = (S_1, \dots, S_n)$ of a multivariate asset, based on a $d = 2n$ dimensional Hawkes process of the form (1), with representation

$$S_1 = N_1 - N_2, \qquad S_2 = N_3 - N_4, \dots, \qquad S_n = N_{d-1} - N_d. \tag{2}$$

In this context, the fact that the S_i take values on $\mathbb{Z}$ accounts for the discreetness of the price formation under a limit order book. If we take $\mu_{2i-1} = \mu_{2i}$ for every $1 \le i \le d$, the process S is centred and the mutually exciting properties of the intensity processes λ_i under (1) allow to reproduce empirically microstructure noise and the Epps effect, as demonstrated in [7]. Microstructure noise – a major stylised fact in high frequency financial data (see *e.g.* [6,4,5,26,27,8]) – is characterised by the property that in microscopic scales,[1] an upward change

[1] When the data are sampled every few seconds or less.