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Stochastic Processes and their Applications 123 (2013) 2500–2521

stochastic processes and their applications

www.elsevier.com/locate/spa

An infinite dimensional convolution theorem with applications to the efficient estimation of the integrated volatility

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Available online 4 April 2013

Abstract

This paper proposes a general approach to obtain asymptotic lower bounds for the estimation of random functionals. The main result is an abstract convolution theorem in a non parametric setting, based on an associated LAMN property. This result is then applied to the estimation of the integrated volatility, or related quantities, of a diffusion process, when the diffusion coefficient depends on an independent Brownian motion.

MSC: primary 62G20; secondary 60F05; 60H05

Keywords: LAMN property; Convolution theorem; Diffusion process

1. Introduction

A fundamental concept in the parametric estimation theory is the notion of *Locally Asymptotically Normal* (LAN) families of distributions, introduced by Le Cam (see Le Cam and Yang [17], Van der Vaart [21]). In particular, this notion permits to establish asymptotic lower bounds for the distribution of any 'regular' estimator of a parameter θ. More precisely, a classical result,

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http://dx.doi.org/10.1016/j.spa.2013.04.004

known as the Hajek convolution theorem, states that the asymptotic distribution of any 'regular' estimator is necessarily a convolution between a Gaussian law and some other law. An advantage of this result is to give a natural way to introduce the notion of efficiency, in the case where the asymptotic distribution reduces to the Gaussian part just mentioned above. In a lot of situations, the LAN property is not satisfied but a more general condition, called *Locally Asymptotically Mixed Normality* (LAMN), can be established. In this latter case, the Hajek convolution theorem can be extended (see Jeganathan [15,16]) and the asymptotic distribution of any 'regular' estimator can be conditionally decomposed as a convolution.

In the LAN situation, some extensions have been done in a non parametric setting by Millar [18], Ibragimov and Kha'sminskii [7] and Golubev [5], but it seems that, up to now, similar results are still unknown in the LAMN case. The aim of this paper is to propose a Hajek type convolution theorem, for the estimation of a random functional, in a LAMN setting. For a random variable F with value in a space B, we consider the general estimation problem of $\Phi(F)$, based on the observation of a random variable with law P_n on a measurable space $(E_n, \mathcal{B}_n)$. The main assumption is that the probability P_n can be decomposed as $P_n(A) = \int_B P_n^f(A) dP^F$, where P^F is the law of the random variable F and $\{P_n^f, f \in B\}$ a statistical experiment, depending on an infinite dimensional parameter f and satisfying the LAMN property. In this Bayesian framework with prior P^F, we establish a convolution theorem which does not require any regularity assumption on the estimator, but requires some regularity on the prior P^F. This convolution theorem permits to define in a rigorous way the notion of asymptotic efficiency for the estimation of random functionals. Moreover, we give some extensions to the estimation of a quantity depending both on the observations and the prior P^F. Such situations occur frequently in practice.

In a second part, we apply our infinite dimensional convolution theorem to the estimation of some functionals of a diffusion process discretely observed. We assume that we observe at times $(t_i^n)_i$ the process X, the solution of

$$X(t) = x_0 + \int_0^t b(X(s))ds + \int_0^t a(X(s), \sigma(s))dW(s),$$

where $(\sigma(t))_t$ is an Itô process, independent of W. This problem can be connected to the preceding abstract framework since we observe a process depending on a random unknown infinite dimensional parameter σ. Our applications concern among others the estimation of quantities which appears in stochastic finance, such as the integrated volatility $\int_0^1 a^2(X(s), \sigma(s))ds$ or some stochastic integrals $\int_0^1 \chi(t, X(t))dX(t)$ related to hedging problems. From our convolution results, we derive explicit lower bounds for the estimation of these quantities based on a discrete sampling of X. Another application deals with the efficiency of discretization schemes such as the Euler scheme.

The paper is organized as follows. In Section 2, we derive an infinite dimensional convolution theorem based on the LAMN property. Section 3 is devoted to the applications to a discretely observed diffusion process. The technical proofs are postponed to the Appendix.

2. Infinite dimensional convolution theorem

2.1. Definitions and notations

Throughout this paper we will consider a real separable Hilbert space H, equipped with the inner product $\langle ., . \rangle$ and the associated norm $\|.\|$, and a subset $B \subset H$. Let I be a linear bounded positive self-adjoint operator on H, then I admits a square-root $I^{1/2}$, such that $I^{1/2} I^{1/2} = I$,