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Ergodicity of observation-driven time series models and consistency of the maximum likelihood estimator

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Abstract

This paper deals with a general class of observation-driven time series models with a special focus on time series of counts. We provide conditions under which there exist strict-sense stationary and ergodic versions of such processes. The consistency of the maximum likelihood estimators is then derived for well-specified and misspecified models.

Keywords: Consistency; Ergodicity; Time series of counts; Maximum likelihood; Observation-driven models; Stationarity

There has recently been a strong renewed interest in developing models for time series of counts which arise in a wide variety of applications: economics, finance, epidemiology, population dynamics, etc. Among the models proposed so far, observation-driven models introduced by Cox [1] play an important role (see [14, Chapter 4] for a comprehensive account and [20] for a recent survey). In time series of counts, the observations are the realizations of some integer-valued distribution (e.g. Poisson, negative binomial, ...) depending on some parameters that drives the dynamic of the model. In this paper, we focus on the so-called observation-driven time series models in which the parameter depends solely on the past observations. Examples of such models include Poisson integer-valued GARCH (INGARCH) (see [7] or [22,8]), Poisson threshold models (see [12]), log-linear Poisson autoregression (see [9]); see also [2,3,16] for other observation-driven models for Poisson counts.

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This paper discusses the theory and inference for a general class of observation-driven models which includes the models introduced above as particular examples. Compared to the approach introduced in [9], our argument is not based on the so-called perturbation technique which seems technically involved and heavily relies on the Poisson assumption. The approach developed by Neumann [16] is more direct but is based on a contraction assumption on the intensity of the Poisson variable which is not satisfied, for example neither in the log-linear Poisson autoregression model nor in the Poisson threshold model. We do not follow the weak dependence approach which as outlined in [5] also implies unnecessary Lipschitz assumptions of the model and does not yield directly a theory for likelihood inference.

Our approach is based on the theory of Markov chains without irreducibility assumption. We first prove the existence of a stationary distribution using the result of [21]. The main difficulty when the Markov chain is not necessarily irreducible consists in proving the uniqueness of the stationary distribution. For that purpose, we extend the delicate argument introduced by Henderson et al. [12] and based on the theory of asymptotically strong Feller Markov chains (see [11]). Our extension introduces a drift term which adds considerable flexibility on the model assumptions and allows to cover the log-linear Poisson autoregression model under assumptions which are weaker than those reported in [9]. We then establish ergodicity for the two-sided stationary version of the process under the sole assumption of existence and uniqueness of the stationary distribution. Finally, we develop the theory of likelihood inference by approximating the conditional likelihood by an appropriately defined stationary version of it, which is shown to converge using classical ergodic theory arguments. Our likelihood inference theory covers both well-specified and misspecified models. We focus on the consistency of the conditional likelihood estimator but the asymptotic normality can also be covered using stationary martingale arguments. Due to space constraints, this will be reported in a forthcoming paper.

The organization of the paper is as follows. Section 1 formulates the model, establishes the existence and uniqueness of the invariant distribution and shows the ergodicity and existence of some moments for the observation process. The maximum likelihood estimates of the parameters and the relevant asymptotic theory are then derived in Section 2. Examples of threshold autoregressive and log Poisson counts are used to illustrate our findings. The proofs are given in Section 3. Finally, Appendices contains general statements about the ergodicity of Markov chains under minimal assumptions which might be of independent interest.

1. Ergodicity of the observation-driven time series model

Let (X, d) be a locally compact, complete and separable metric space and denote by $\mathcal{X}$ the associated Borel sigma-field. Let $(\mathsf{Y}, \mathcal{Y})$ be a measurable space, H a Markov kernel from $(\mathsf{X}, \mathcal{X})$ to $(\mathsf{Y}, \mathcal{Y})$ and $(x, y) \mapsto f_y(x)$ a measurable function from $(\mathsf{X} \times \mathsf{Y}, \mathcal{X} \otimes \mathcal{Y})$ to $(\mathsf{X}, \mathcal{X})$.

Definition 1. An *observation-driven time series model on* $\mathbb{N}$ is a stochastic process $\{(X_n, Y_n), n \in \mathbb{N}\}$ on $\mathsf{X} \times \mathsf{Y}$ satisfying the following recursions: for all $k \in \mathbb{N}$,

$$\begin{aligned} &Y_{k+1} | \mathcal{F}_k \sim H(X_k; \cdot), \\ &X_{k+1} = f_{Y_{k+1}}(X_k), \end{aligned} \tag{1}$$

where $\mathcal{F}_k = \sigma(X_\ell, Y_\ell; \ell \le k, \ell \in \mathbb{N})$. Similarly, $\{(X_n, Y_n), n \in \mathbb{Z}\}$ is an *observation-driven time series model on* $\mathbb{Z}$ if the previous recursion holds for all $k \in \mathbb{Z}$ with $\mathcal{F}_k = \sigma(X_\ell, Y_\ell; \ell \le k, \ell \in \mathbb{Z})$.