Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 124 (2014) 81–100

stochastic processes and their applications

www.elsevier.com/locate/spa

Integral representation of martingales motivated by the problem of endogenous completeness in financial economics

Dmitry Kramkov[a,b,*], Silviu Predoiu[c]

[a] *Carnegie Mellon University, Department of Mathematical Sciences, 5000 Forbes Avenue, Pittsburgh, PA, 15213-3890, USA*

[b] *University of Oxford, Department of Mathematical Sciences, 5000 Forbes Avenue, Pittsburgh, PA, 15213-3890, USA*

[c] *Citigroup, NY, USA*

Received 19 November 2012; received in revised form 15 April 2013; accepted 17 June 2013
Available online 13 July 2013

Abstract

Let $\mathbb{Q}$ and $\mathbb{P}$ be equivalent probability measures and let ψ be a J-dimensional vector of random variables such that $\frac{d\mathbb{Q}}{d\mathbb{P}}$ and ψ are defined in terms of a weak solution X to a d-dimensional stochastic differential equation. Motivated by the problem of *endogenous completeness* in financial economics we present conditions which guarantee that every local martingale under $\mathbb{Q}$ is a stochastic integral with respect to the J-dimensional martingale $S_t \triangleq \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t]$. While the drift $b = b(t, x)$ and the volatility $\sigma = \sigma(t, x)$ coefficients for X need to have only minimal regularity properties with respect to x, they are assumed to be analytic functions with respect to t. We provide a counter-example showing that this t-analyticity assumption for σ cannot be removed.

MSC: primary 60G44; 91B51; 91G99; secondary 35K15; 35K90

Keywords: Integral representation; Martingales; Diffusion; Parabolic equations; Analytic semigroups; Real analytic functions; Krylov–Ito formula; Dynamic completeness; Equilibrium

* Corresponding author at: Carnegie Mellon University, Department of Mathematical Sciences, 5000 Forbes Avenue, Pittsburgh, PA, 15213-3890, USA. Tel.: +1 412268 5912.

E-mail address: kramkov@cmu.edu (D. Kramkov).

0304-4149/$ - see front matter
http://dx.doi.org/10.1016/j.spa.2013.06.017

1. Introduction

Let $(\Omega, \mathcal{F}_1, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$ be a complete filtered probability space, $\mathbb{Q}$ be an equivalent probability measure, and $S = (S_t^j)$ be a J-dimensional martingale under $\mathbb{Q}$. It is often important to know whether every local martingale $M = (M_t)$ under $\mathbb{Q}$ admits an integral representation with respect to S, that is,

$$M_t = M_0 + \int_0^t H_u dS_u, \quad t \in [0, 1], \tag{1.1}$$

for some predictable S-integrable process $H = (H_t^j)$. For instance, in mathematical finance, which is the topic of a particular interest to us, the existence of such a martingale representation corresponds to the *completeness* of the market model driven by stock prices S, see Harrison and Pliska [7].

A general answer is given in Jacod [9, Section XI.1(a)]. Jacod's theorem states that the integral representation property holds if and only if $\mathbb{Q}$ is the unique equivalent martingale measure for S. In mathematical finance this result is sometimes referred to as the 2nd fundamental theorem of asset pricing.

In many applications, the process S is defined in a *forward form*, in terms of its predictable characteristics under $\mathbb{P}$. The density process Z of a martingale measure $\mathbb{Q}$ for S is then constructed through the use of the Girsanov theorem and its generalizations, see Jacod and Shiryaev [10]. The verification of the existence of integral representations for all $\mathbb{Q}$-martingales under S is often straightforward. For example, if S is a diffusion process under $\mathbb{P}$ with the drift vector-process $b = (b_t)$ and the volatility matrix-process $\sigma = (\sigma_t)$, then such a representation exists if and only if σ has full rank $d\mathbb{P} \times dt$ almost surely.

In this paper we assume that both S and Z are described in a *backward form*, through their terminal values. Given random variables $\xi > 0$ and $\psi = (\psi^j)_{j=1,\dots,J}$ they are defined as

$$Z_1 \triangleq \frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \frac{\xi}{\mathbb{E}[\xi]},$$
$$S_t \triangleq \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t], \quad t \in [0, 1].$$

We are looking for (easily verifiable) conditions on ξ and ψ guaranteeing the integral representation of all $\mathbb{Q}$-martingales with respect to S.

Our work is motivated by the problem of *endogenous completeness* in continuous-time financial economics which naturally arises in the construction of Radner equilibrium, see Anderson and Raimondo [1], Hugonnier, Malamud, and Trubowitz [8], and Riedel and Herzberg [18], and in the study of the equilibrium-based price impact models, see Bank and Kramkov [2] and German [6]. Here ξ is an equilibrium state price density, usually defined implicitly by a fixed point argument, and $\psi = (\psi^j)$ is the random vector of the cumulative discounted dividends for traded stocks. The term “endogenous” is used because the stock prices S are *computed* as an output of equilibrium. A similar problem also arises in the verification of the completeness of markets where, in addition to stocks, one can also trade options, see Davis and Obłój [5].

We focus on the case when ξ and ψ are defined in terms of a weak solution X to a d-dimensional stochastic differential equation. With respect to x the coefficients of this equation satisfy the classical conditions guaranteeing weak existence and uniqueness: the drift vector $b(t, \cdot)$ is measurable and bounded and the volatility matrix $\sigma(t, \cdot)$ is uniformly continuous and