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Stochastic Processes and their Applications 124 (2014) 101–122

stochastic processes and their applications

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# On signed measure valued solutions of stochastic evolution equations

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Received 18 February 2012; received in revised form 11 July 2013; accepted 11 July 2013
Available online 22 July 2013

**Abstract**

We study existence, uniqueness and mass conservation of signed measure valued solutions of a class of stochastic evolution equations with respect to the Wiener sheet, including as particular cases the stochastic versions of the regularized two-dimensional Navier–Stokes equations in vorticity form introduced by Kotelenez.


*MSC:* primary 76D06; 60G57; 60H15; secondary 60H05; 60J60

*Keywords:* Signed measure; Stochastic evolution equation; McKean–Vlasov; Wiener sheet

## 1. Introduction

Measure-valued stochastic processes arise as mathematical descriptors for the limiting behavior of many characteristic parameters used for modeling complex evolutions in the natural sciences. Genetic drift, bacterial spread, fluid dynamics, heat conduction and chemical reactions are but a few examples of challenging problems for which a good mathematical understanding can be reached by first building an appropriate interacting particle system and then looking at it

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0304-4149/$ - see front matter 
http://dx.doi.org/10.1016/j.spa.2013.07.003

through the lens of the associated (measure valued) empirical process. When properly rescaled, the processes give rise to measure valued limits solving stochastic evolution equations which are themselves of great interest. The study of the existence and uniqueness of measure valued solutions to stochastic evolution equations really started with Dawson [6] and the subject has grown rapidly since then. For a nice review of measure valued processes, see [7].

A broad class of such equations, the so-called stochastic McKean–Vlasov equations, e.g., [8], are probability valued if their starting point is. This conservation of the initial mass turns out to be quite easy to show in this case, by way of a basic completeness argument which remains valid for many other examples in the literature. Caution must prevail, however, against a common misconception which recurs unfortunately too often, to the effect that an interacting particle system displaying no creation or annihilation of particles at any time, necessarily gives rise in its scaling limits to stochastic evolutions obeying this mass conservation property. While seemingly sensible, such a statement requires proof.

For a general treatment of the important and challenging family of stochastic Navier–Stokes equations in vorticity form, such as those appearing in [14], it is necessary to look for solutions in the space of signed measures. Mainly because the space of signed measures is not complete for the usual metrics compatible with the topology of weak convergence, it is much harder to study existence, uniqueness and/or mass conservation of signed measure valued solutions of stochastic evolution equations.

In fact, because of the incompleteness, many of the results presented in the literature on signed measure valued solutions are either false or have only been provided with a false proof. This is the case for example in [21,14,1], where the space of signed measures was assumed to be complete. Other examples of incorrect proofs include [15–17]. These articles will be discussed later.

In this paper, we study existence and uniqueness of signed measure valued solutions of a class of stochastic evolution equations with respect to the Wiener sheet, including as particular cases the stochastic versions of the regularized two-dimensional Navier–Stokes equations in vorticity form introduced by Kotelenez. These stochastic evolution equations can be seen as weak versions of equations of the form

$$d\chi_t = L^*_{\chi_t}\chi_t - \nabla(\Gamma\chi_t)dW,$$

where $L^*$ is the (formal) adjoint of a diffusion operator with coefficients depending on $\chi$ and $W$ is the Wiener sheet. Other weak versions of these equations appear for example in [2]. Also, additional references to similar equations are given in Section 5 where we discuss the relationship with two-dimensional regularized Navier–Stokes equations in vorticity form.

The problem is described in Section 2, where appropriate spaces of measures are defined. Next, in Section 3, using a particle representation, signed measure valued solutions are shown to exist when the initial signed measure has finite support. The existence is then shown to hold for general initial conditions. Uniqueness and mass conservation are shown to hold in Section 4, using fixed point arguments and duality. Finally, in Section 5, we revisit some of the results appearing in the literature concerning two-dimensional Navier–Stokes equations in vorticity form.

A commendable attempt at resolving all these issues can be found in [19], where the authors show the existence of solutions to a large class of nonlinear stochastic partial differential equations encompassing our own. Their techniques also allow them to prove the uniqueness of solutions, but only for those starting measures with square integrable densities with respect to the Lebesgue measure. Our constructions yield the existence and uniqueness of solutions for all starting signed measures.