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Stochastic Processes and their Applications 124 (2014) 220–234

stochastic processes and their applications

www.elsevier.com/locate/spa

Harnack inequality on configuration spaces: The coupling approach and a unified treatment

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Received 22 October 2012; received in revised form 1 April 2013; accepted 18 July 2013
Available online 13 August 2013

Abstract

In this paper, we establish the dimension-free Harnack inequality on configuration spaces by using the coupling argument. Furthermore, a unified treatment is also used to prove the equivalence between the Harnack inequality on configuration space and that on the corresponding base space under a very mild condition.

MSC: 60J45; 58J65; 60J60

Keywords: Harnack inequality; Configuration space; Second quantization; Coupling

1. Introduction

The second quantization of a Markov semigroup, which can be realized by an independent particle system on the configuration space (cf. [16, (5.3)]), is a fundamental model in the study of infinite-dimensional analysis. Second quantization is a powerful tool used in quantum field theory for describing the many-particle systems. We refer to [10,15] for physical background of the second quantization semigroup. A key point for the study of this model is to characterize the second quantization semigroup by using properties of the base process. For known results concerning particle systems on configuration spaces, we refer to [13,14,23,5] for functional inequalities and exponential convergence, [8] for the Feller and strong Feller properties, and [24] for small time behaviors. In this paper, we aim to establish the dimension-free Harnack inequality in the sense of [17] on configuration spaces. See e.g. [22] and the references therein for applications of the dimension-free Harnack inequality.

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0304-4149/$ - see front matter
http://dx.doi.org/10.1016/j.spa.2013.07.008

Let us first introduce the basic framework. Let $(M, \mathscr{F})$ be a measurable space, and let $P(x, \mathrm{d}y)$ be a transition probability on M. Then

$$Pf(x) := \int_M f(y)\, P(x, \mathrm{d}y), \quad x \in M,\ f \in \mathscr{B}_b(M)$$

gives rise to a Markov operator P. Here and in what follows, $\mathscr{B}_b(M)$ (resp. $\mathscr{B}_b^+(M)$) denotes the set of all bounded measurable (resp. bounded non-negative measurable) functions on M. Denote by δ_x the Dirac measure concentrated at $x \in M$. Consider the infinite configuration space

$$\Gamma := \left\{ \gamma = \sum_{i=1}^{\infty} \delta_{x_i};\ x_i \in M \right\}$$

equipped with the σ-field induced by $\{\gamma \mapsto \gamma(A);\ A \in \mathscr{F}\}$. In many cases, one may be restricted to the locally finite configuration space

$$\Gamma_0 := \big\{ \gamma \in \Gamma;\ \gamma(K) < \infty \text{ for compact } K \subset M \big\},$$

on which the induced σ-field coincides with the Borel σ-field of the vague topology. See [1] for more details on analysis and geometry on configuration spaces. Let $\mathscr{B}_b(\Gamma)$ be the set of all bounded measurable functions on Γ, and $\mathscr{B}_b^+(\Gamma)$ the set of all non-negative elements in $\mathscr{B}_b(\Gamma)$. The Markov operator P^Γ considered in this paper is

$$P^\Gamma F(\gamma) := \int_{M^{\mathbb{N}}} F\left(\sum_{i=1}^{\infty} \delta_{z_i} \right) \prod_{i=1}^{\infty} P(x_i, \mathrm{d}z_i), \qquad \gamma = \sum_{i=1}^{\infty} \delta_{x_i},\ F \in \mathscr{B}_b(\Gamma).$$

The central purpose of this paper is to discuss the dimension-free Harnack inequality for P^Γ. It is well-known that the coupling argument, developed in [2,20], is quite efficient for proving a Harnack inequality (see also [9,11,22]). Therefore, it is natural for us to investigate the Harnack inequality for P^Γ via the coupling approach. Since the underlying space Γ is infinite-dimensional, on the other hand, we can use techniques in infinite-dimensional analysis to handle this problem. A unified treatment will be presented (see Section 3 below).

In order to formulate the Harnack inequality for P^Γ, we need a distance-like function on $\Gamma \times \Gamma$. Let

$$I : M^{\mathbb{N}} \to \Gamma, \qquad (x_i)_{i \ge 1} \mapsto \sum_{i=1}^{\infty} \delta_{x_i}.$$

For any non-negative function φ on $M \times M$, define

$$\varphi^\Gamma(\gamma, \eta) = \inf \left\{ \sum_{i=1}^{\infty} \varphi(x_i, y_i);\ (x_i)_{i \ge 1} \in I^{-1}(\gamma),\ (y_i)_{i \ge 1} \in I^{-1}(\eta) \right\}, \quad \gamma, \eta \in \Gamma. \tag{1.1}$$

The organization of this paper is as follows. The Harnack inequality for P^Γ is established in Section 2 via a coupling approach. We first present a general result, and then as an application diffusion processes on the configuration space over a Riemannian manifold are considered. The results are also applied to study the strong Feller property, hyper-bounded property, and entropy-cost inequality. In Section 3, a unified treatment is used to prove that under a very mild condition P^Γ satisfies the Harnack inequality iff so does P. Finally, we present a fundamental property of φ^Γ in Section 4.