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Global uniform boundary Harnack principle with explicit decay rate and its application

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Abstract

In this paper, we consider a large class of subordinate Brownian motions X via subordinators with Laplace exponents which are complete Bernstein functions satisfying some mild scaling conditions at zero and at infinity. We first discuss how such conditions govern the behavior of the subordinator and the corresponding subordinate Brownian motion for both large and small time and space. Then we establish a global uniform boundary Harnack principle in (unbounded) open sets for the subordinate Brownian motion. When the open set satisfies the interior and exterior ball conditions with radius $R > 0$, we get a global uniform boundary Harnack principle with explicit decay rate. Our boundary Harnack principle is global in the sense that it holds for all $R > 0$ and the comparison constant does not depend on R, and it is uniform in the sense that it holds for all balls with radii $r \leq R$ and the comparison constant depends neither on D nor on r. As an application, we give sharp two-sided estimates for the transition densities and Green functions of such subordinate Brownian motions in the half-space.

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1. Introduction

The study of potential theory of discontinuous Lévy processes in $\mathbb{R}^d$ revolves around several fundamental questions such as sharp heat kernel and Green function estimates, exit time estimates and Poisson kernel estimates, Harnack and boundary Harnack principles for non-negative harmonic functions. One can roughly divide these studies into two categories: those on a bounded set and those on an unbounded set. For the former, it is the local behavior of the process that matters, while for the latter both local and global behaviors are important. The processes investigated in these studies are usually described in two ways: either the process is given explicitly through its characteristic exponent (such as the case of a symmetric stable process, a relativistically stable process, sum of two independent stable processes, etc.), or some conditions on the characteristic exponent are given. In the situation when one is interested in the potential theory on bounded sets, conditions imposed on the characteristic exponent govern the small time–small space (i.e., local) behavior of the process. Let us be more precise and describe in some detail one such condition and some of the results in the literature.

Let $S = (S_t)_{t\geq 0}$ be a subordinator (that is, an increasing Lévy process satisfying $S_0 = 0$) with Laplace exponent ϕ, and let $W = (W_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^d$, $d \geq 1$, independent of S with

$$\mathbb{E}_x\left[e^{i\xi(W_t-W_0)}\right] = e^{-t|\xi|^2}, \quad \xi \in \mathbb{R}^d,\ t > 0.$$

The process $X = (X_t)_{t\geq 0}$ defined by $X_t := W(S_t)$ is called a subordinate Brownian motion. It is a rotationally invariant Lévy process in $\mathbb{R}^d$ with characteristic exponent $\phi(|\xi|^2)$. The function ϕ is a Bernstein function. Let us introduce the following upper and lower scaling conditions:

(H1): There exist constants $0 < \delta_1 \leq \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1\left(\frac{R}{r}\right)^{\delta_1} \leq \frac{\phi(R)}{\phi(r)} \leq a_2\left(\frac{R}{r}\right)^{\delta_2}, \quad 1 \leq r \leq R. \tag{1.1}$$

It follows from the definitions in [2, pp. 65 and 68] and [2, Proposition 2.2.1] that (1.1) is equivalent to saying that ϕ is in the class OR of O-regularly varying functions at ∞ with Matuszewska indices contained in (0, 1). The advantage of the formulation above is that we can provide more direct proofs for some of the results below. (1.1) is a condition on the asymptotic behavior of ϕ at infinity and it governs the behavior of the subordinator S for small time and small space, which, in turn, implies the small time–small space behavior of the corresponding subordinate Brownian motion X. Very recently it has been shown in [19] (see also [15]) that if (H1) holds and ϕ is a complete Bernstein function, then the uniform boundary Harnack principle is true and various exit time and Poisson kernel estimates hold. Further, sharp two-sided Green function estimates for bounded $C^{1,1}$ open sets are given in [15]. The statements of these results usually take the following form: for some $R > 0$, there exists a constant $c = c(R) > 0$ (also depending on the process X) such that some quantities involving $r \in (0, R)$ can be estimated by expressions involving the constant c. The point is that although the constant c is uniform for small $r \in (0, R)$, it does depend on R, meaning that the result is local. It would be of interest to obtain a global and uniform version of such results, namely with the constant depending neither on R nor on the open set itself. This would facilitate the study of potential theory on unbounded sets. In order to accomplish this goal, it is clear that the assumption (H1) (or some similar condition) will not suffice, and that one needs additional assumptions that govern the behavior of the process for large time and large space.