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Upper escape rate of Markov chains on weighted graphs

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Abstract

We obtain an upper escape rate function for a continuous time minimal symmetric Markov chain defined on a locally finite weighted graph. This upper rate function, which has the same form as the manifold setting, is given in terms of the volume growth with respect to an adapted path metric. Our approach also gives a weak form of Folz's theorem on the conservativeness as a consequence.

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1. Introduction

1.1. Brownian motion case

The celebrated Khintchine's law of the iterated logarithm states that, for the Brownian motion $(B_t)_{t\geq 0}$ on $\mathbb{R}$,

$$\limsup_{t\to\infty} \frac{|B_t|}{\sqrt{2t\log\log t}} = 1, \quad \text{a.s.}$$

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In particular, we see that for the function $R(t) = \sqrt{(2+\varepsilon)t\log\log t}$ ($\varepsilon > 0$),

$$\mathbb{P}_0\left(|B_t| \le R(t) \text{ for all sufficiently large } t\right) = 1.$$

Such a function $R(t)$ is called an *upper escape rate function* of the Brownian motion.

To extend this notion to more general Markov processes, we have to take care. Let (V, d) be a locally compact separable metric space and μ a positive Radon measure on V with full support. Let $V_\infty = V \cup \{\infty\}$ be the one point compactification of (V, d). If (V, d) is already compact, then ∞ is adjoined as an isolated point. Let $\mathcal{M} = \left(\Omega, (X_t)_{t\ge 0}, \{\mathbb{P}_x\}_{x\in V\cup\{\infty\}}, \{\mathcal{F}_t\}_{t\ge 0}, \infty, \zeta\right)$ be a μ-symmetric Hunt process on V. Here the sample space Ω is taken to be the space of right-continuous functions $\omega : [0,\infty] \to V_\infty$ such that:

(1) ω has a left limit $\omega(t-) \in V_\infty$ for any $t \in (0,\infty)$;
(2) $\omega(t) = \infty$ for any $t \ge \zeta := \inf\{s \ge 0 : \omega(s) = \infty\}$ and $\omega(\infty) = \infty$.

The random variable X_t is defined as $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$. The random variable ζ is called the lifetime of the process $\mathcal{M}$, which can be finite. We will feel free to use some further properties of Hunt processes as presented in [8, Section A.2].

Definition 1.1. Fix a reference point $\bar{x} \in V$. A nonnegative increasing function $R(t)$ is called an upper rate function for the process $\mathcal{M}$, if there exists a random time $T < \zeta$ such that

$$\mathbb{P}_{\bar{x}}\left(d(X_t, \bar{x}) \le R(t) \text{ for all } T \le t < \zeta\right) = 1.$$

Remark 1.2. We do not extend the distance d to V_∞, so $d(X_t, \bar{x})$ has no definition if $t \ge \zeta$.

The upper rate function gives a quantitative criterion for the conservativeness. Indeed, the existence of it implies that $\mathbb{P}_x(\zeta < \infty) = 0$ for all $x \in V$.

Many authors studied the escape rate for the Brownian motion on a complete Riemannian manifold. Grigor'yan [11] initiated the study of upper rate functions in terms of the volume growth of Riemannian manifolds. Grigor'yan and Hsu [13] obtained a sharp form of upper rate functions for Cartan–Hadamard manifolds. Recently, Hsu and Qin [14] obtained a sharp result in full generality.

Theorem 1.3 (*Hsu and Qin [14]*)**.** *Let M be a complete Riemannian manifold and fix $\bar{x} \in M$. Let $B(r)$ be the geodesic ball on M of radius r and centered at $\bar{x}$. Assume that*

$$\int^{\infty} \frac{r\,dr}{\log \operatorname{vol}(B(r))} = \infty, \tag{1.1}$$

where $\int^\infty$ means that we only care about the singularity at ∞. Define

$$\psi(R) = \int_6^R \frac{r\,dr}{\log \operatorname{vol}(B(r)) + \log\log r}.$$

Then there is a constant $C > 0$ such that $C\psi^{-1}(Ct)$ is an upper rate function of the Brownian motion $(X_t)_{t\ge 0}$ on M.

Remark 1.4. As observed in [14], (1.1) is equivalent to that $\lim_{R\to\infty}\psi(R) = \infty$.