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Asymptotic behavior of central order statistics from stationary processes

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Abstract

In this paper, we show that central order statistics from strictly stationary and ergodic sequences are strongly consistent estimators of population quantiles provided that the quantiles are unique. We generalize this result to strictly stationary but not necessarily ergodic sequences. We also describe three types of possible asymptotic behavior of central order statistics in the case when the corresponding population quantile is not unique. We give applications of the presented results to linear processes with both absolutely continuous and discrete innovations.

MSC: 60F15; 60G10; 62G30; 62G20

Keywords: Central order statistics; Stationary processes; Linear processes; Quantiles; Conditional quantiles; Almost sure convergence

1. Introduction

Let $(X_n, n \geq 1)$ be a sequence of random variables (rv's) with common cumulative distribution function (cdf) F, and $X_{1:n} \leq \cdots \leq X_{n:n}$ be the order statistics corresponding to the sample $(X_1, \ldots, X_n)$. If $(k_n, n \geq 1)$ is a sequence of positive integers such that $k_n \leq n$ for all n and $k_n/n \to \lambda \in (0, 1)$ as $n \to \infty$ then $X_{k_n:n}, n \geq 1$, are referred to as central order statistics.

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Various asymptotic properties of $X_{k_n:n}$ as $n \to \infty$ are known in the literature. The best described case is the case where F is sufficiently smooth in a neighborhood of the λth population quantile. Assuming that $F(\gamma_\lambda) = \lambda$, F has at least two derivatives in a neighborhood of γ_λ, F'' is bounded in the neighborhood and $F'(\gamma_\lambda) > 0$, Bahadur [2] gave an asymptotic almost sure representation for sample quantiles of independent and identically distributed (iid) rv's via the empirical distribution function. In fact, Bahadur's [2] proof shows more, namely that this representation holds not only for sample quantiles but for any central order statistics $X_{k_n:n}$ such that $n^{3/4}(\log n)^{-1}(k_n/n - \lambda) \to 0$ as $n \to \infty$. This representation not only gives an insight into the probabilistic structure of sample quantiles but also is a useful tool — it can serve, for example, to show in the iid case that under suitable conditions sample quantiles are asymptotically normal and that the law of iterated logarithm holds for sample quantiles. Bahadur's representation has been improved and generalized by many authors. Its refinements and other versions of proof in the iid setting were given by, among others, Kiefer [21], Ghosh [15], de Haan and Taconis-Haantjes [7], Einmahl [11] and Knight [22]. Its extensions to dependent rv's can be found in [26] for m-dependent processes, in [10] for autoregressive processes, in [27] for ϕ-mixing processes, in [19] for short-range dependent linear processes, in [20] for long-range dependent linear processes and in [33] for linear and some nonlinear processes.

Smirnov [29] derived the asymptotic distributional theory for central order statistics without imposing any restrictions on F. Assuming $(k_n/n - \lambda)\sqrt{n} \to 0$ and that observations are iid, he identified the four possible types of limiting laws as $n \to \infty$ of $X_{k_n:n}$, suitably standardized when necessary, and characterized the domain of attraction for each type. Balkema and de Haan [4] pointed out that if the condition $(k_n/n - \lambda)\sqrt{n} \to 0$ is dropped and we only require that $k_n \to \infty$, $n - k_n \to \infty$ and $k_n/n \to \lambda$ then in the iid setting for any $\lambda \in [0, 1]$ any desired cdf can appear as the limiting cdf of suitably standardized $X_{k_n:n}$.

Smirnov [29] also showed that if the rv's $X_n, n \geq 1$, are iid with cdf F such that $\underline{\gamma}_\lambda = \overline{\gamma}_\lambda$, where

$$\underline{\gamma}_\lambda := \inf\{x \in \mathbb{R} : F(x) \geq \lambda\} \quad \text{and} \quad \overline{\gamma}_\lambda := \sup\{x \in \mathbb{R} : F(x) \leq \lambda\}, \tag{1.1}$$

then $X_{k_n:n}$ is a strongly consistent estimator of the λth population quantile for any choice of the sequence $(k_n, n \geq 1)$ provided only that $1 \leq k_n \leq n$ for all n and $k_n/n \to \lambda \in (0, 1)$. A shorter proof of this property but under stronger assumptions can be found in [12, p. 195]. If instead $\underline{\gamma}_\lambda \neq \overline{\gamma}_\lambda$, then, as shown by Feldman and Tucker [13], $X_{k_n:n}$ does not converge almost surely and oscillates infinitely often or converges almost surely to $\underline{\gamma}_\lambda$ or to $\overline{\gamma}_\lambda$ depending on the way of the convergence of the sequence $(k_n/n, n \geq 1)$ to λ.

Our objective in this paper is to extend the results mentioned in the latter paragraph to strictly stationary sequences. In Section 2, we show that central order statistics from strictly stationary and ergodic sequences are strongly consistent estimators of population quantiles provided that the quantiles are unique and then give an extension of this property to strictly stationary but not necessarily ergodic sequences. Next, in Section 3, we complete the results of Feldman and Tucker [13] keeping the iid assumption while in Section 4 we establish general conditions under which these results hold also for strictly stationary sequences. Finally in Section 5, we discuss a special case of strictly stationary sequences, namely linear processes, and show how results of the previous sections can be applied to linear processes with absolutely continuous innovations and with discrete innovations. In the Appendix we give, for ease of reference, some known results used in our proofs.

Throughout the paper we make use of the following notation. Unless otherwise stated, the rv's $X_n, n \geq 1$, exist in a probability space $(\Omega, \mathcal{F}, P)$. $\mathbb{R}$ and $\mathbb{Z}$ represent the sets of real numbers and