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# A Sobolev space theory for parabolic stochastic PDEs driven by Lévy processes on $C^1$-domains

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**Abstract**

In this paper we study parabolic stochastic partial differential equations (SPDEs) driven by Lévy processes defined on $\mathbf{R}^d$, $\mathbf{R}^d_+$ and bounded $C^1$-domains. The coefficients of the equations are random functions depending on time and space variables. Existence and uniqueness results are proved in (weighted) Sobolev spaces, and $L_p$-estimates and various properties of solutions are also obtained. The number of derivatives of the solutions can be any real number, in particular it can be negative or fractional.


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*Keywords:* Stochastic partial differential equations; Lévy processes; Sobolev spaces; $L_p$-theory

## 1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{\mathcal{F}_t, t \geq 0\}$ be an increasing filtration of right continuous $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$, each of which contains all $(\mathcal{F}, P)$-null sets. Assume that on $\Omega$ we are given independent one-dimensional Wiener processes $B^1_t, B^2_t, \dots$ and (pure jump) Lévy processes $Z^1_t, Z^2_t, \dots$ relative to $\{\mathcal{F}_t, t \geq 0\}$.

Many aspects of the $L_2$-theory of SPDEs of the type

$$du = \left(a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu + f\right)dt + \left(\sigma^{ik}u_{x^i} + \mu^ku + g^k\right)dB^k_t \tag{1.1}$$

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were well investigated long time ago by Pardoux [28], Krylov and Rozovskii [23], Flandoli [6], Da Prato and Zabczyk [5] and Brzezniak [3] and few others. Also see Gyöngy [9] for an $L_2$-theory of SPDEs driven by semi-martingales. The semi-group approaches, the monotonicity arguments and integration by parts have been main tools in the $L_2$-theory.

An $L_p$-theory for Eq. (1.1) on $\mathbf{R}^d$ was first introduced by Krylov in [17,18]. Later, a weighed $L_p$-theory for Eq. (1.1) on domains was constructed in [21,22] (on half space) and [10,11,14] (on bounded $C^1$-domains). As mentioned in [18] there are many advantages of the $L_p$-theory over the $L_2$-theory. The embedding $W_p^n \subset C^{n-d/p}$ (if $n - d/p > 0$) shows that the larger $p$ is the better regularity of solutions we get. The regularity of solutions can be used, for instance, to show how fast the numerical solutions converge to true solutions. Also, the $L_p$-theory allows one to get the required regularity of solutions under weaker smoothness conditions of the coefficients. For example, in the framework of $L_p$-theory the solutions are continuously differentiable in $x$ if the coefficients are merely continuous in $x$ and $p > d$; however in the framework of the $L_2$-theory to guarantee that the solutions are continuously differentiable in $x$, it is required that the coefficients are more than $(d-2)/2$ times continuously differentiable in $x$ (see [18] for details).

The main goal of this article is to extend existing results on Eq. (1.1) and present an $L_p$-theory of the equation

$$
\begin{aligned}
du &= \left(a^{ij}u_{x^ix^j} + b^i u_{x^i} + cu + f\right)dt \\
&\quad + \left(\sigma^{ik}u_{x^i} + \mu^k u + g^k\right)dB_t^k + \left(\bar{\sigma}^{ik}u_{x^i} + \bar{\mu}^k u + h^k\right)dZ_t^k \qquad (1.2)
\end{aligned}
$$

defined on $\mathbf{R}^d$, $\mathbf{R}^d_+$ and bounded $C^1$ domains. Here, the summation with respect to repeated indices $i, j, k$ is assumed, $i$ and $j$ go from 1 to $d$, and $k$ runs through $\{1, 2, \ldots\}$. The coefficients $a^{ij}, b^i, c, \sigma^{ik}, \bar{\sigma}^{ik}, \mu^k$ and $\bar{\mu}^k$ are random functions depending on $(t, x)$. We say $u$ is a solution of (1.2) if $u$ satisfies (1.2) in the sense of Definition 2.6. The issue regarding the convergence of the series of stochastic integrals in (1.2) is discussed in Remark 2.5(ii).

We refer the readers e.g. to [26,29,30] for many applications of Eq. (1.2). We only mention that Eq. (1.2) with random coefficients appears, for instance, in nonlinear filtering problems (estimations of the signal by observing it when it is mixed with noises), and many other types of equations, for example, driven by (Lévy) space–time white noises can be written in the form of (1.2) (see [18]).

We remark that if $\sigma^{ik} = \mu^k = g^k = 0$, an $L_p$-theory for Eq. (1.2) on $\mathbf{R}^d$, Theorem 2.23, was already introduced in [4]. However, [4] does not contain various properties of solutions which are essential for the study of $L_p$-theory of SPDEs on domains. For example, inequalities (2.17), (2.30) and (2.35), which were not proved in [4], are main tools for us to study SPDEs on the half space. Furthermore, in [4] the regularity condition on $h$ is not sharp, and it is improved in this article.

We also refer to [13,27] for the $L_p$-theory of stochastic integro-differential equations on $\mathbf{R}^d$ driven by jump processes.

As in [10,11,14,21,22], we use Sobolev spaces with weights to deal with SPDEs on domains. This is because the Hölder space approach does not allow one to obtain results of reasonable generality, and Sobolev spaces without weights turn out to be trivially inappropriate (see [21,22] for details). We only mention that unless certain compatibility conditions are fulfilled (cf. [6]), the derivatives of solutions blow up near the boundary of domains, and this blow-up can be controlled with the help of appropriate weights.