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Stochastic Processes and their Applications 124 (2014) 505–521
stochastic processes and their applications
www.elsevier.com/locate/spa

A strong law of large numbers for super-stable processes

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Received 24 December 2012; received in revised form 11 August 2013; accepted 28 August 2013
Available online 6 September 2013

Abstract

Let ℓ be Lebesgue measure and $X = (X_t, t \geq 0; P_\mu)$ be a supercritical, super-stable process corresponding to the operator $-(-\Delta)^{\alpha/2}u + \beta u - \eta u^2$ on $\mathbb{R}^d$ with constants $\beta, \eta > 0$ and $\alpha \in (0, 2]$. Put $\hat{W}_t(\theta) = e^{(|\theta|^\alpha - \beta)t} X_t(e^{-i\theta \cdot})$, which for each *small* θ is an a.s. convergent complex-valued martingale with limit $\hat{W}(\theta)$ say. We establish for any starting finite measure μ satisfying $\int_{\mathbb{R}^d} |x|\mu(\mathrm{d}x) < \infty$ that $\frac{t^{d/\alpha} X_t}{e^{\beta t}} \to c_\alpha \hat{W}(0)\, \ell\ P_\mu$-a.s. in a topology, termed the shallow topology, strictly stronger than the vague topology yet weaker than the weak topology, where $c_\alpha > 0$ is a known constant. This result can be thought of as an extension to a class of superprocesses of Watanabe's strong law of large numbers for branching Markov processes.

MSC: 60J68; 60F15; 60G57

Keywords: Super-stable process; Super-Brownian motion; Strong law of large numbers; Fourier transform; Vague convergence; Probability measures

1. Introduction

We use $M_F(\mathbb{R}^d)$ to denote the set of finite measures on $\mathbb{R}^d$. We use $\mu(f)$ to denote $\int f \mathrm{d}\mu$ for a measure μ and integrable function f. It is clear that $\mu(D) = \mu(I_D)$, where I_D is the indicator function of D. Let $C_c(\mathbb{R}^d)$ denote the set of continuous functions on $\mathbb{R}^d$ with compact support.

In 1967, Watanabe [28] first discussed the strong law of large numbers for branching Brownian motion. Let $(X_t, t \geq 0; P_x)$ be a branching Brownian motion on $\mathbb{R}^d$ $(d \geq 1)$ starting

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0304-4149/$ - see front matter
http://dx.doi.org/10.1016/j.spa.2013.08.009

from a single point $x \in \mathbb{R}^d$ and corresponding to the operator

$$\frac{1}{2}\triangle u + a(F(u) - u),$$

where a is a positive constant and $F(s) := \sum_{n=0}^{\infty} p_n s^n$, $s \geq 0$, is the generating function of the offspring distribution $\{p_n, n \geq 0\}$. By explicitly using the Gaussian density, Watanabe [28] proved in the supercritical case, i.e. $\beta := a(F'(1) - 1) > 0$, that under the condition $\sum_{n=0}^{\infty} n^2 p_n < \infty$, it follows that

$$\frac{X_t}{e^{\beta t} t^{-d/2}} \to (2\pi)^{-d/2} \ell \cdot W, \quad P_x\text{-a.s.} \tag{1}$$

as $t \to \infty$ in the sense of vague convergence, where ℓ is the Lebesgue measure on $\mathbb{R}^d$ and W is the limit of the martingale $W_t := e^{-\beta t} X_t(1)$. Later, based on the ideas in [28], Biggins [2] proved a strong law of large numbers for discrete-time branching random walk.

Suppose $(X_t, t \geq 0; P_\mu)$ is a super-Brownian motion on $\mathbb{R}^d$, $d \geq 1$, corresponding to the operator $\frac{1}{2}\triangle u + \beta u - \eta u^2$, where $\beta > 0$ and $\eta > 0$ are positive constants, and starting from $\mu \in M_F(\mathbb{R}^d)$. For background on measure-valued processes see Dawson [7]. Then, it seems that Englander [11] was the first to discuss the law of large numbers for the supercritical super-Brownian motion $(X_t, t \geq 0; P_\mu)$. It was proved in [11] that for any $f \in C_c(\mathbb{R}^d)$,

$$\frac{X_t(f)}{e^{\beta t} t^{-d/2}} \to (2\pi)^{-d/2} \ell(f) \cdot W, \quad \text{in } P_\mu\text{-probability}, \tag{2}$$

where W is the limit of the martingale $W_t := e^{-\beta t} X_t(1)$. More recently, Wang [27] improved the convergence in (2) from "in probability" to "P_μ-a.s." in the special case that $\mu = \delta_x$, $x \in \mathbb{R}^d$, by combining the Fourier analysis used [28] and the uniform convergence method for martingales used in [2]. Wang's proof depends on the specific density of Brownian motion and the compact support property of super-Brownian motion starting from a compactly supported measure. For more path properties of super-Brownian motion, see Dawson, Iscoe and Perkins [8], Dawson and Perkins [10], and Perkins [24,25]. But, α-stable processes ($\alpha \in (0, 2)$) do not have specific density expressions. More critically, for any $t > 0$, the support of X_t, the super-stable process with index $\alpha \in (0, 2)$, is the whole space $\mathbb{R}^d$ even when the starting measure μ has compact support (see Dawson and Perkins [10] or Perkins [25]). Therefore, the methods in Wang [27] do not transfer over to general $\mu \in M_F(\mathbb{R}^d)$ nor to a super-stable process with index $\alpha \in (0, 2)$.

Note that both for branching Brownian motion and super-Brownian motion, the mean of X_t is described by the linear operator $\frac{1}{2}\triangle + \beta$ on $\mathbb{R}^d$. The denominator $e^{\beta t} t^{-d/2}$ in (1) and (2) is exactly the growth rate of $e^{\beta t} S_t^{\frac{1}{2}\Delta}$, the semigroup corresponding to $\frac{1}{2}\triangle + \beta$ on $\mathbb{R}^d$, as $t \to \infty$. In our more general α-stable case, corresponding to the operator $-(-\triangle)^{\frac{\alpha}{2}} + \beta$, it will again turn out that the correct scaling, $e^{\beta t} t^{-d/\alpha}$, is dictated by the growth rate of $e^{\beta t} S_t^{\triangle^\alpha}$, the semigroup corresponding to $-(-\triangle)^{\frac{\alpha}{2}} + \beta$.

If $\frac{1}{2}\triangle$ is replaced by a diffusion operator L with spatially dependent coefficients or more general operator and β is spatially dependent, the strong (or weak) law of large numbers for branching diffusion (or more general branching Hunt processes) and superdiffusion have been investigated recently by many papers. See [1,6] for branching diffusion, [12] for branching Hunt processes, and [5,11,14,15,23] (with general branching mechanism) for superdiffusions. In all of these papers, the mean of the process grows pure exponentially as $e^{\lambda_c t}$ with some positive