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A mixed-step algorithm for the approximation of the stationary regime of a diffusion

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Abstract

In some recent papers, some procedures based on some weighted empirical measures related to decreasing-step Euler schemes have been investigated to approximate the stationary regime of a diffusion (possibly with jumps) for a class of functionals of the process. This method is efficient but needs the computation of the function at each step. To reduce the complexity of the procedure (especially for functionals), we propose in this paper to study a new scheme, called the *mixed-step scheme*, where we only keep some regularly time-spaced values of the Euler scheme. Our main result is that, when the coefficients of the diffusion are smooth enough, this alternative does not change the order of the rate of convergence of the procedure. We also investigate a Richardson–Romberg method to speed up the convergence and show that the variance of the original algorithm can be preserved under a uniqueness assumption for the invariant distribution of the "duplicated" diffusion, condition which is extensively discussed in the paper. Finally, we conclude by giving sufficient "asymptotic confluence" conditions for the existence of a smooth solution to a discrete version of the associated Poisson equation, condition which is required to ensure the rate of convergence results.

MSC: 60G10; 60J60; 65C05; 65D15; 60F05

Keywords: Stochastic differential equation; Stationary process; Steady regime; Ergodic diffusion; Central Limit Theorem; Euler scheme; Poisson equation; Richardson–Romberg extrapolation

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1. Introduction

In a series of papers [10,13,19,18,16,17] going back to [9], have been investigated the properties of a Euler scheme with decreasing step as a tool for the numerical approximation of the stationary regime of a diffusion, possibly with jumps, satisfying stability (mean reverting) conditions. The purpose of the present paper is to propose and investigate a variant of the original procedure sharing the similar properties in terms of convergence and rate but with a lower complexity, especially in its functional version, i.e. when trying to compute the expectation of a functional of the process (over a finite time interval $[0, T]$) with respect to the stationary distribution of the process.

In this paper we will focus on the case of Brownian diffusions. We consider an $\mathbb{R}^d$-valued diffusion process $(X_t)_{t\geq 0}$ solution to

$$(\text{SDE}) \;\equiv dX_t = b(X_t)dt + \sigma(X_t)dW_t, \tag{1.1}$$

where $(W_t)_{t\geq 0}$ is a q-dimensional standard Brownian motion (SBM) and the coefficients b and σ are *Lipschitz continuous* functions from $\mathbb{R}^d$ to $\mathbb{R}^d$ and $\mathbb{M}_{d,q}$ respectively (where $\mathbb{M}_{d,q}$ denotes the set of $d \times q$-matrices). Under these assumptions, strong existence and uniqueness hold for the SDE starting from any $\mathbb{R}^d$-valued r.v. independent of W and $(X_t)_{t\geq 0}$ is a homogeneous Markov process with semi-group $(P_t)_{t\geq 0}$. We will denote by $\mathbb{P}_\mu$, the distribution of the whole process $(X_t)_{t\geq 0}$ (supported by the set $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$) when starting from X_0 with distribution μ. We also assume throughout the paper that $(X_t)_{t\geq 0}$ has a unique invariant distribution ν. Except in the one-dimensional case, ν cannot be made explicit and the numerical computation of ν or $\mathbb{P}_\nu$ (which, in particular, is fundamental, to estimate the asymptotic behavior of ergodic processes) then requires some specific numerical methods.

Let us briefly describe the discrete and continuous time Euler schemes with decreasing step resulting from the time discretization of the diffusion $(X_t)_{t\geq 0}$. First we introduce a non-decreasing sequence $(\Gamma_n)_{n\geq 1}$ of discretization times starting from $\Gamma_0 = 0$ and we assume that the step sequence, defined as its increments, by $\gamma_n := \Gamma_n - \Gamma_{n-1},\ n \geq 1$, is *non-increasing* and satisfies

$$\lim_{n\to+\infty} \gamma_n = 0 \quad \text{and} \quad \Gamma_n = \sum_{k=1}^{n} \gamma_k \xrightarrow{n\to+\infty} +\infty. \tag{1.2}$$

The discrete time Euler scheme $(\bar{X}_{\Gamma_n})_{n\geq 0}$ (with Brownian increments) is recursively defined at discretization times Γ_n by $\bar{X}_0 = x_0$ and

$$\bar{X}_{\Gamma_{n+1}} = \bar{X}_{\Gamma_n} + \gamma_{n+1} b(\bar{X}_{\Gamma_n}) + \sigma(\bar{X}_{\Gamma_{n+1}})(W_{\Gamma_{n+1}} - W_{\Gamma_n}). \tag{1.3}$$

If we introduce the notation

$$\underline{t} = \Gamma_{N(t)} \quad \text{with } N(t) = \min\{n \geq 0, \Gamma_{n+1} > t\} \tag{1.4}$$

so that $\underline{t} = \Gamma_k$ if and only if $t \in [\Gamma_k, \Gamma_{k+1})$, the stepwise constant Euler scheme also reads

$$\forall t \in \mathbb{R}_+, \quad \bar{X}_t = \bar{X}_{\underline{t}}.$$

The idea at the origin of [9] was to make the guess, mimicking the pointwise ergodic theorem, that the weighted empirical measure

$$\nu_n(\omega, dx) = \frac{1}{\Gamma_n} \sum_{k=1}^{n} \gamma_k \delta_{\bar{X}_{\Gamma_{k-1}}(\omega)}(dx) \tag{1.5}$$