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Moment boundedness of linear stochastic delay differential equations with distributed delay☆

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Highlights

- This paper presents the characteristic function for moment stability of the linear stochastic delay differential equations with distributed delay.
- From the characteristic function, we obtain sufficient conditions for the second moment to be bounded or unbounded.

Abstract

This paper studies the moment boundedness of solutions of linear stochastic delay differential equations with distributed delay. For a linear stochastic delay differential equation, the first moment stability is known to be identical to that of the corresponding deterministic delay differential equation. However, boundedness of the second moment is complicated and depends on the stochastic terms. In this paper, the characteristic function of the equation is obtained through techniques of the Laplace transform. From the characteristic equation, sufficient conditions for the second moment to be bounded or unbounded are proposed.

MSC: 34K06; 34K50

Keywords: Stochastic delay differential equation; Distributed delay; Moment boundedness

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1. Introduction

Time delays are known to be involved in many processes in biology, chemistry, physics, engineering, *etc.*, and delay differential equations are widely used in describing these processes. Delay differential equations have been extensively developed in the past several decades (see [1,2,7]). Furthermore, stochastic perturbations are often introduced into these deterministic systems in order to describe the effects of fluctuations in the real environment, and thus yield stochastic delay differential equations. Mathematically, stochastic delay differential equations were first introduced by Itô and Nisio in the 1960s [8] in which the existence and uniqueness of the solutions have been investigated. In the last several decades, numerous studies have been developed toward the study of stochastic delay differential equations, such as stochastic stability, Lyapunov functional method, Lyapunov exponent, stochastic flow, invariant measure, invariant manifold, numerical approximation and attraction *etc.* (see [3–6,9,10,12,11,14,15,17,16,18, 20–22] and the references therein). However, many basic issues remain unsolved even for a simple linear equation with constant coefficients.

In this paper, we study the following linear stochastic differential equation with distributed delay

$$\begin{aligned} dx(t) = {} & \left(ax(t) + b\int_0^{+\infty} K(s)x(t-s)ds\right)dt \\ & + \left(\sigma_0 + \sigma_1 x(t) + \sigma_2\int_0^{+\infty} K(s)x(t-s)ds\right)dW_t. \end{aligned} \tag{1.1}$$

Here a, b and $\sigma_i (i = 0, 1, 2)$ are constants, W_t is a one dimensional Wiener process, and $K(s)$ represents the density function of the delay s. In this study, we always assume Itô interpretation for the stochastic integral. This paper studies the moment boundedness of the solutions of (1.1). Particularly, this paper gives the characteristic function of the equation, through which sufficient conditions for the second moment to be bounded or unbounded are obtained.

Despite the simplicity of (1.1), which is a linear equation with constant coefficients, current understanding for how the stability and moment boundedness depend on the equation coefficients is still incomplete. Most of known results are obtained through the method of Lyapunov functional. The Lyapunov functional method is useful for investigating the stability of differential equations, and has been well developed for delay differential equations [7], stochastic differential equations [15], and stochastic delay differential equations [10,12,14,15]. The Lyapunov functional method can usually give sufficient conditions for the stability of stochastic delay differential equations. For general results one can refer to the Razumikhin-type theorems on the exponential stability for the stochastic functional differential equation [15, Chapter 5]. However, these results often depend on the method of how the Lyapunov functional is constructed and are incomplete, not always applicable for all parameter regions. For example, sufficient conditions for the pth moment stability of the following stochastic differential delay equation

$$dx(t) = (ax(t) + bx(t-\tau))\,dt + (\sigma_1 x(t) + \sigma_2 x(t-\tau))\,dW_t, \tag{1.2}$$

can be obtained when $a < 0$, but not for $a > 0$ [15, Example 6.9 in Chapter 5].

In 2007, Lei and Mackey [13] introduced the method of Laplace transform to study the stability and moment boundedness of Eq. (1.1) with discrete delay ($K(s) = \delta(s-1)$). In this particular case, the characteristic equation was proposed, which yields a sufficient (and is also necessary if not of the critical situation) condition for the boundedness of the second moment (see Theorem 3.6 in [13]). This result gives a complete description for the second moment stability