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On the solution of general impulse control problems using superharmonic functions

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Abstract

In this paper, a characterization of the solution of impulse control problems in terms of superharmonic functions is given. In a general Markovian framework, the value function of the impulse control problem is shown to be the minimal function in a convex set of superharmonic functions. This characterization also leads to optimal impulse control strategies and can be seen as the corresponding characterization to the description of the value function for optimal stopping problems as a smallest superharmonic majorant of the reward function. The results are illustrated with examples from different fields, including multiple stopping and optimal switching problems.

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Keywords: Impulse control strategies; Superharmonic functions; General Markov processes

1. Introduction

Stochastic control techniques play a major role in many fields of applied probability. In particular, the developments in mathematical finance have stimulated the activities in this branch of control theory in the last decades. Many of these approaches have the disadvantage that they lead to non-realizable optimal strategies since these strategies consist of interventions at each time instant in a continuous time model. The right mathematical framework to consider discrete interventions in a continuous time model is given by impulse control problems.

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Impulse control problems have been studied for decades. It seems to be impossible to give an overview over all fields of application and all different variants that have been used. We only want to mention finance, e.g. cash management and portfolio optimization, see [18,25], optimal forest management, see [29,2] and the references therein, and control of an exchange rate by the Central Bank, see [22,7]. Most of these articles are based on the seminal work developed in [6], which still turns out to be the main reference for theoretical results in this field. For underlying diffusion process under some further assumptions, the value function is proved to be a solution of a corresponding quasi-variational inequality, that also characterizes the optimal strategy. A more recent overview over results for jump–diffusions is given in [24], see also [18] for a survey with focus on financial applications.

On the other hand, it is known that there is a strong connection between impulse control problems and problems of optimal stopping. Under certain conditions, the value function of the impulse control problem can be found as the limit of a sequence of value functions for associated optimal stopping problems, see [24, Chapter 7]. Moreover, the value function of the impulse control problem can be characterized as a solution to an implicit problem of optimal stopping, where implicit means that the reward function in the optimal stopping problem contains this value function itself, see [18].

For Markovian problems of optimal stopping, the most flexible and valuable approach – both from a theoretical and practical point of view – seems to be the superharmonic characterization of the value function; more precisely, under minimal condition, the value function is the smallest superharmonic function majorizing the reward function. This characterization goes back to Dynkin [16] and turned out to be the right formulation for most such problems. For an explicit solution, this approach can be translated into free-boundary problems, which can be solved in many problems of interest. An excellent overview over recent developments in this field is given in the monograph [26]. One of main advantages of considering superharmonic functions (instead of, e.g. using a formulation using quasi-variational inequalities) is that regularity conditions can often be stated in a more natural way from a stochastic point of view.

One of the consequences of this superharmonic characterization is that optimal stopping problems for an underlying one-dimensional diffusion process can be solved explicitly in many situations of interest, since the superharmonic functions turn out to be transformed concave functions, see [14,5], or [11] for recent treatments. Therefore, one can say that optimal stopping of one-dimensional diffusion processes is well-understood. Inspired by these result, in the last years, different authors considered special classes of impulse control problems for an underlying one-dimensional diffusion processes, and obtained a solution in terms of superharmonic (resp. excessive) functions, see [1,3,17]. One of the main advantages of these approaches is that they work in a very general setting without strong regularity assumptions on the problem, that are often needed for applying alternative approaches.

The question arises whether there is also a general characterization of the value function of an impulse control problem as the smallest function in a set of superharmonic functions, as for optimal stopping problems. The aim of this article is to consider the impulse control problem from a purely superharmonic point of view to use the well-known advantages for optimal stopping problems also for impulse control problems. This is carried out in a very general Markovian setting in the following section. The main results are Theorems 2.3 and 2.6, that give a characterization of the value function as well as the existence and description of an optimal strategy under very general conditions: under natural assumptions, (for the problem without integral term) the value function of an impulse control can be characterized as the smallest r-superharmonic function h with $Mh \leq h$, where M denotes the maximum operator. This can be