Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 124 (2014) 961–983

stochastic processes and their applications

www.elsevier.com/locate/spa

Zero-sum risk-sensitive stochastic games on a countable state space

Arnab Basu[a,*], Mrinal Kanti Ghosh[b]

[a] *Quantitative Methods and Information Systems Area, Indian Institute of Management, Bangalore, Karnataka, 560076, India*

[b] *Department of Mathematics, Indian Institute of Science, Karnataka, 560012, India*

Received 24 April 2013; received in revised form 10 August 2013; accepted 18 September 2013
Available online 29 September 2013

Abstract

Infinite horizon discounted-cost and ergodic-cost risk-sensitive zero-sum stochastic games for controlled Markov chains with countably many states are analyzed. Upper and lower values for these games are established. The existence of value and saddle-point equilibria in the class of Markov strategies is proved for the discounted-cost game. The existence of value and saddle-point equilibria in the class of stationary strategies is proved under the uniform ergodicity condition for the ergodic-cost game. The value of the ergodic-cost game happens to be the product of the inverse of the risk-sensitivity factor and the logarithm of the common Perron–Frobenius eigenvalue of the associated controlled nonlinear kernels.

Keywords: Risk-sensitive stochastic games; Exponential discounted and ergodic costs; Shapley equations

1. Introduction

We study risk-sensitive zero-sum stochastic games on the infinite time horizon on a countable state space. The risk-sensitive cost criterion plays an important role in many applications including mathematical finance (see, e.g., Nagai [33]). In this criterion one investigates ‘exponential of integral’ cost which takes into account the attitude of the controller with respect to risk. The study of this kind of cost criteria was first initiated in Bellman [4, p. 329] for the finite-state space case. In Howard and Matheson [26], an in-depth analysis was carried out for the finite-state space case where each controlled chain is irreducible and aperiodic. This was

* Corresponding author. Tel.: +91 9742221302.
E-mail addresses: arnab.basu@iimb.ernet.in (A. Basu), mkg@math.iisc.ernet.in (M.K. Ghosh).

0304-4149/$ - see front matter
http://dx.doi.org/10.1016/j.spa.2013.09.009

extended in Rothblum [34] to the general non-irreducible finite-state space case. In the past two decades, there has been a renewed interest in this type of cost criteria as, when the 'risk factor' is strictly positive, the use of the exponential reduces the possibility of rare but devastating large excursions of the state process. Though this criterion has been studied extensively in the literature of Markov decision processes (see, e.g., Borkar and Meyn [11], Cavazos-Cadena and Fernández-Gaucherand [12], Cavazos-Cadena and Hernández-Hernández [13], Di Masi and Stettner [14–16], Fleming and Hernández-Hernández [21], Fleming and McEneaney [23], Hernández-Hernández and Marcus [24], Whittle [37,38]), and in the literature of stochastic optimal control (see, e.g., Bensoussan et al. [6], Bensoussan and Nagai [7], Biswas et al. [9], Fleming and McEneaney [23], Menaldi and Robin [31], Nagai [33], Runolfsson [35]), the corresponding results on stochastic games are limited (see, e.g., Basar [2], Basu and Ghosh [3], El-Karoui and Hamadene [18], Fleming and Hernández-Hernández [22], Jacobson [27], James et al. [28], Klompstra [29]). The general Linear–Exponential–Gaussian (LEG) control problem for discrete time with perfect state observation is treated in Jacobson [27] where an equivalence of this with deterministic zero-sum quadratic-cost games was shown. This was extended to studying the Nash equilibrium for a two-person discrete-time nonzero-sum game with quadratic–exponential cost criteria in Klompstra [29], the analogue of the Linear–Exponential–Quadratic–Gaussian (LEQG) control problem studied in Whittle [36,37]. The papers of Basar [2] and El-Karoui and Hamadene [18] deal with stochastic differential games on the finite time horizon. The only relevant literature for stochastic differential games on the infinite horizon for the multiplicative cost criterion can be found in Basu and Ghosh [3]. In James et al. [28], the finite-horizon risk-sensitive stochastic optimal control problem for discrete-time nonlinear systems was studied and its relation to a deterministic partially observed dynamic game was established.

In this paper, we extend the results of Di Masi and Stettner [14], Hernández-Hernández and Marcus [24] to zero-sum stochastic games on the infinite time horizon with both discounted and ergodic cost criteria. A zero-sum risk-sensitive stochastic game is relevant in the analysis of worst-case scenarios, e.g., in financial applications when a risk-averse investor is trying to minimize his long-term portfolio loss against the market which, by default, is antagonistic and hence the maximizer in this case. Thus the minimizer chooses the risk-aversion parameter $\theta > 0$ and minimizes his expected risk-sensitive costs over the infinite horizon. This justifies the choice of $\theta > 0$ as opposed to $\theta < 0$ in which case the minimizer would have been risk-seeking. The maximizer is not risk-seeking but essentially antagonistic to the minimizer. For additive cost criteria, such a problem is standard in worst-case analysis. In our paper, we have extended this idea to the multiplicative cost. The problem addressed in this paper has been studied in the case of controlled diffusions in Basu and Ghosh [3]. But it should be emphasized that the analysis and techniques for stochastic differential games are entirely different from the corresponding problems in discrete time.

Under certain assumptions, we establish the existence of values and saddle-point equilibria for both criteria. We also show that the value of the ergodic-cost game is shown to be the product of the inverse of the risk-sensitivity factor and the logarithm of the common Perron–Frobenius eigenvalue of the associated controlled nonlinear kernels. We obtain our results by studying the corresponding Shapley equations (also called Isaacs' equations for differential games). It should be noted here that in some of the existing literature in this domain (see, e.g., Cavazos-Cadena and Fernández-Gaucherand [12] and Hernández and Marcus [24]), the 'risk factor' is assumed to be sufficiently small. Our analysis does not make any such assumptions on the 'risk factor'.

The rest of our paper is structured as follows. Section 2 deals with the description of the problem. The discounted cost criterion is studied in Section 3. Section 4 deals with the ergodic