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A monotonicity property for random walk in a partially random environment

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Abstract

We prove a law of large numbers for random walks in certain kinds of i.i.d. random environments in $\mathbb{Z}^d$ that is an extension of a result of Bolthausen et al. (2003) [4]. We use this result, along with the lace expansion for self-interacting random walks, to prove a monotonicity result for the first coordinate of the speed of the random walk under some strong assumptions on the distribution of the environment.

MSC: 60K37; 82C41

Keywords: Random walk in a random environment; Monotonicity of speed; Lace expansion

1. Introduction

First studied in the pioneering work of Solomon and others in the mid-1970s to early 1980s, random walks in a random environment (RWRE) has enjoyed a revival in recent times as a number of interesting results have been obtained. Many of these results relate to laws of large numbers and invariance principles for i.i.d. random environments that are uniformly elliptic (all nearest-neighbor transition probabilities are bounded away from zero). While the behavior of the one-dimensional RWRE is quite well understood, the understanding is much less complete for RWRE in higher dimensions, and in particular, for non-ballistic RWRE. Several special

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classes of RWRE are amenable to analysis in general dimensions, such as the random walk among random conductances, random walks in balanced random environments or Dirichlet random environments, random walks in random environments which are small perturbations of a deterministic environment, etc. (see e.g. [1,3,5,11] and the references therein). Another class is that of a random walk in a *partially random environment*, introduced by Bolthausen et al. in [4]. They established laws of large numbers and central limit theorems for a RWRE in dimensions $d = d_0 + d_1$, where $d_0 \geq 1$ is the number of coordinates in which the environment is random, and where the projection of the walk onto the remaining $d_1 \geq 5$ coordinates is a deterministic symmetric random walk.

In this paper we consider monotonicity properties of the speed for random walks in partially random environments (RWpRE) that are similar to those considered in [4]. Such properties have not been extensively studied in the literature. The few results on the monotonicity of the speed that we are aware of include the work of Holmes and Salisbury [9] where monotonicity of the speed (when it exists) is proved for all environments that take only two possible values via a coupling argument, and that of Sabot [11] and Fribergh [6], where asymptotic expansions for the speed are derived for random walks in random environments that are small perturbations of a simple random walk with drift. There has also been recent progress in the study of the monotonicity of the speed as a function of the bias for a biased random walk on supercritical percolation clusters [7] and Galton–Watson trees [2]. Our main result is a monotonicity result for the first coordinate of the speed, under some special assumptions on the distribution of the partially random environment. For example when at each site either the left or right step in the first coordinate direction is not available, we prove that the first coordinate of the speed is monotone increasing in the probability that the right step is available. Our proof consists of two steps. We first extend a result of [4] to show that the (non-random) speed exists almost surely for the class of RWpRW under consideration here. We then establish the desired monotonicity by analyzing an expansion formula for the speed derived in [14] using lace expansion techniques, which is valid for all annealed RWRE, but is most useful in the case $d_1 \gg d_0$ when one has good control (in terms of finite random walk Green's functions) over the terms in the expansion.

Let $\mathcal{M}_1(\mathbb{Z}^d)$ be the space of probability kernels on $\mathbb{Z}^d$, and more generally for $c > 0$ let $\mathcal{M}_c(\mathbb{Z}^d)$ denote the space of kernels on $\mathbb{Z}^d$ with total mass c. Given a family of probability kernels $\omega := (\omega_{x,m}(\cdot))_{x\in\mathbb{Z}^d, m\in\mathbb{N}} \in \mathcal{M}_1(\mathbb{Z}^d)^{\mathbb{Z}^d\times\mathbb{N}}$ which we call a *cookie environment*, the law of a random walk $(X_n)_{n\geq 0}$ in the cookie environment ω starting at $X_0 = x$, denoted by $P_{x,\omega}$, is defined as follows. Under $P_{x,\omega}$, the walk evolves conditionally on its history via the transition probabilities $P_{x,\omega}(X_n = X_{n-1} + u|(X_i)_{0\leq i\leq n-1}) = \omega_{X_{n-1},\ell_{n-1}(X_{n-1})}(u)$, for all $n \in \mathbb{N}$, where $\ell_n(y) = \sum_{k=0}^{n} 1_{\{X_k=y\}}$ is the number of visits to y up to time n. In words, upon the mth visit to x, the walk sees the environment $\omega_{x,m}$ and makes a jump accordingly. We will consider the case where ω is random, and the cookie environment at different points in space, $(\omega_{x,\cdot}(\cdot))_{x\in\mathbb{Z}^d}$, which are i.i.d. with a common law $\mu \in \mathcal{M}_1(\mathcal{M}_1(\mathbb{Z}^d)^{\mathbb{N}})$. The measure $P_{x,\omega}$ is called the *quenched* law. When we average the quenched law of $(X_n)_{n\geq 0}$ with respect to the cookie environment ω, we obtain the so-called *annealed* (or more accurately the *averaged*) law

$$P_x := \mathbb{P} \times P_{x,\omega},$$

where $\mathbb{P} := \mu^{\otimes\mathbb{Z}^d}$ denotes the law of ω in the product space $\Omega := (\mathcal{M}_1(\mathbb{Z}^d))^{\mathbb{Z}^d\times\mathbb{N}}$.

The random walk model described above is sometimes called a multi-excited random walk (in a random cookie environment). When we restrict the cookie environment to environments that are constant in m, i.e. $\omega_{x,m}(\cdot) \equiv \omega_x(\cdot)$ for every x and m, we obtain the more often studied RWRE