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Stochastic algorithms for computing means of probability measures

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Abstract

Consider a probability measure μ supported by a regular geodesic ball in a manifold. For any $p \geq 1$ we define a stochastic algorithm which converges almost surely to the p-mean e_p of μ. Assuming furthermore that the functional to minimize is regular around e_p, we prove that a natural renormalization of the inhomogeneous Markov chain converges in law into an inhomogeneous diffusion process. We give an explicit expression of this process, as well as its local characteristic.

Keywords: Mean; Barycenter; Probability measure; Riemannian geometry; Convexity; Geodesic ball; Markov chain; Convergence in law; Invariance principle

1. Introduction

Consider a set of points $\{x_1, \ldots, x_n\}$ in an Euclidean space E with metric d. The geometric barycenter e_2 of this set of points is the unique point minimizing the mean square distance to these points, i.e.

$$e_2 = \arg\min_{x \in E} \frac{1}{n} \sum_{i=1}^{n} d^2(x, x_i).$$

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It is equal to the standard mean $e_2 = \frac{1}{n}\sum_{i=1}^{n} x_i$ and is the most common estimator in statistics. However it is sensitive to outliers, and it is natural to replace power 2 by p for some $p \in [1, 2)$. This leads to the definition of p-means: for $p \geq 1$, a minimizer of the functional

$$H_p : \begin{array}{ccc} E & \to & \mathbb{R}_+ \\ x & \mapsto & \displaystyle\frac{1}{n}\sum_{i=1}^{n} d^p(x, x_i) \end{array}$$

is called a p-mean of the set of points $\{x_1, \ldots, x_n\}$. When $p = 1$, e_1 is the median of the set of points and is very often used in robust statistics. In many applications, p-means with some $p \in (1, 2)$ give the best compromise. It is well known that in dimension 1, the median of a set of real numbers may not be uniquely defined. This is however an exceptional case: the p-mean of a set of points is uniquely defined as soon as $p = 1$ and the points are not aligned or as $p > 1$. In these cases, uniqueness is due to the strict convexity of the functional H_p.

The notion of p-mean is naturally extended to probability measures on Riemannian manifolds. Let μ be a probability measure on a Riemannian manifold M with distance ρ. For any $p \geq 1$, a p-mean of μ is a minimizer of the functional

$$H_p : \begin{array}{ccc} M & \to & \mathbb{R}_+ \\ x & \mapsto & \displaystyle\int_M \rho^p(x, y)\mu(dy). \end{array} \tag{1.1}$$

It should be stressed that unlike the Euclidean case, the functional H_p may not be convex (if $p \geq 2$) and the p-mean may not be uniquely defined. In the case $p = 2$, we obtain the so-called Riemannian barycenter or Karcher mean of the probability measure μ. This has been extensively studied, see e.g. [8–10,5,16,2], where questions of existence, uniqueness, stability, relation with martingales in manifolds, behavior when measures are pushed by stochastic flows have been considered. In the general case $p \geq 1$, Afsari [1] proved existence and uniqueness of p-means on "small" geodesic balls. More precisely, let $\mathrm{inj}(M)$ be the injectivity radius of M and $\alpha^2 > 0$ an upper bound for the sectional curvatures in M. Existence and uniqueness of the p-mean in ensured as soon as the support of the probability measure μ is contained in a convex compact K_μ of a geodesic ball $B(a, r)$ with radius

$$r < r_{\alpha,p} \quad \text{with } r_{\alpha,p} = \begin{cases} \dfrac{1}{2}\min\left\{\mathrm{inj}(M), \dfrac{\pi}{2\alpha}\right\} & \text{if } p \in [1, 2) \\ \dfrac{1}{2}\min\left\{\mathrm{inj}(M), \dfrac{\pi}{\alpha}\right\} & \text{if } p \in [2, \infty). \end{cases} \tag{1.2}$$

The case $p \geq 2$ gives rise to additional difficulties since the functional H_p to minimize is not necessarily convex any more, due to the fact that we can have $r > \frac{\pi}{4\alpha}$.

Provided existence and uniqueness of the p-mean, the question of its practical determination and computation arises naturally. In the Euclidean setting and when $p = 1$, the problem of finding the median e_1 of a set of points is known as the Fermat–Weber problem and numerous algorithms have been designed to solve it. A first one was proposed by Weiszfeld in [18] and was then extended by Fletcher et al. in [6] to cover the case of sufficiently small domains in Riemannian manifolds with nonnegative curvature. A complete generalization to manifolds with positive or negative curvature, including existence and uniqueness results (under some convexity conditions in positive curvature), has been given by one of the authors in [19]. In the case $p = 2$,