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A BSDE approach to stochastic differential games with incomplete information

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Abstract

We consider a two-player zero-sum stochastic differential game in which one of the players has a private information on the game. Both players observe each other, so that the non-informed player can try to guess his missing information. Our aim is to quantify the amount of information the informed player has to reveal in order to play optimally: to do so, we show that the value function of this zero-sum game can be rewritten as a minimization problem over some martingale measures with a payoff given by the solution of a backward stochastic differential equation.

MSC: 93E05; 91A05; 90C39; 60G44; 49N70

Keywords: Stochastic differential games; Backward stochastic differential equations; Dynamic programming; Viscosity solutions

1. Introduction

In this paper we consider a two player zero-sum game, where the underlying dynamics are given by a diffusion with controlled drift but uncontrolled (non-degenerate) volatility. The game can take place in I different scenarios for the running cost and the terminal outcome as in a classical stochastic differential game. Before the game starts one scenario is picked with the probability $p = (p_i)_{i\in\{1,\ldots,I\}} \in \Delta(I)$. The information is transmitted only to Player 1. So at the beginning he knows in which scenario he is playing, while Player 2 only knows the probability

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p. It is assumed that both players observe the actions of the other one, so Player 2 might infer from the actions of his opponent in which scenario the game is actually played.

It has been proved by Cardaliaguet and Rainer in [7] that this game has a value. To investigate the game under the perspective of information transmission we establish an alternative representation of this value. We achieve this by directly modeling the amount of information the informed player reveals during the game. To that end we enlarge the canonical Wiener space to a space which carries besides a Brownian motion, càdlàg martingales with values in $\Delta(I)$. These martingales can be interpreted as possible beliefs of the uninformed player, i.e. the probability in which scenario the game is played in according to his information at time t.

The very same ansatz has been used in the case of deterministic differential games by Cardaliaguet and Rainer in [6], while the original idea of the so called a posteriori martingale can already be found in the classical work of Aumann and Maschler (see [2]). Bearing in mind the ideas of Hamadène and Lepeltier [13] we show that the value of our game can be represented by minimizing the solution of a backward stochastic differential equation (BSDE) with respect to possible beliefs of the uninformed player.

A cornerstone in the investigation of stochastic differential games has been laid by Fleming and Souganidis in [12] who extend the results of Evans and Souganidis [11] to a stochastic framework. Therein it is shown that under Isaacs condition the value function of a stochastic differential game is given as the unique viscosity solution of a Hamilton–Jacobi–Isaacs (HJI) equation.

The theory of BSDE, which was originally developed by Peng [17] for stochastic control theory, has been introduced to stochastic differential games by Hamadène and Lepeltier [13] and Hamadène et al. [14]. The former results have been extended to cost functionals defined by controlled BSDEs in [3], where the admissible control processes are allowed to depend on events occurring before the beginning of the game.

The study of games with incomplete information has its starting point in the pioneering work of Aumann and Maschler (see [2] and references given therein). The extension to stochastic differential games has been given in [7]. The proof is accomplished introducing the notion of dual viscosity solutions to the HJI equation of a usual stochastic differential game, where the probability p just appears as an additional parameter. A different unique characterization via the viscosity solution of the HJI equation with an obstacle in the form of a convexity constraint in p is given in [5]. We use this latter characterization in order to prove our main representation result.

The outline of the paper is as follows. In Section 2 we describe the game and restate the results of [7,5] which build the basis for our investigation. In Section 3 we give our main theorem and derive the optimal behavior for the informed player under some smoothness condition. The whole Section 4 is devoted to the proof of the main theorem, while in the Appendix we summarize extensions to classical BSDE results, which are needed in our case.

2. Setup

2.1. Formal description of the game

Let $\mathcal{C}([0,T];\mathbb{R}^d)$ be the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, which are constant on $(-\infty,0]$ and on $[T,+\infty)$. We denote by $B_s(\omega_B)=\omega_B(s)$ the coordinate mapping on $\mathcal{C}([0,T];\mathbb{R}^d)$ and define $\mathcal{H}=(\mathcal{H}_s)$ as the filtration generated by $s\mapsto B_s$. We denote $\Omega_t=\{\omega\in\mathcal{C}([t,T];\mathbb{R}^d)\}$ and $\mathcal{H}_{t,s}$ the σ-algebra generated by paths up to time s in Ω_t. Furthermore we provide $\mathcal{C}([0,T];\mathbb{R}^d)$ with the Wiener measure $\mathbb{P}^0$ on $(\mathcal{H}_s)$ and we consider the respective filtrations augmented by $\mathbb{P}^0$ nullsets without changing the notation.