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The multifractal nature of Boltzmann processes

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Abstract

We consider the spatially homogeneous Boltzmann equation for (true) hard and moderately soft potentials. We study the pathwise properties of the stochastic process $(V_t)_{t\geq 0}$, which describes the time evolution of the velocity of a typical particle. We show that this process is almost surely multifractal and compute its spectrum of singularities. For hard potentials, we also compute the multifractal spectrum of the position process $(X_t)_{t\geq 0}$.
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1. Introduction

The Boltzmann equation is the main model of kinetic theory. It describes the time evolution of the density $f_t(x, v)$ of particles with position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time $t \geq 0$, in a gas of particles interacting through binary collisions. In the special case where the gas is initially spatially homogeneous, this property propagates with time, and $f_t(x, v)$ does not depend on x. We refer to the books by Cercignani [8] and Villani [22] for many details on the physical and mathematical theory of this equation, see also the review paper by Alexandre [1].

Tanaka gave in [20] a probabilistic interpretation of the case of Maxwellian molecules: he constructed a Markov process $(V_t)_{t\geq 0}$, solution to a Poisson-driven stochastic differential equation, and such that the law of V_t is f_t for all $t \geq 0$. Such a process $(V_t)_{t\geq 0}$ has a richer

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structure than the Boltzmann equation, since it contains some information on the history of particles. Physically, $(V_t)_{t\geq 0}$ is interpreted as the time-evolution of the velocity of a typical particle. Fournier and Méléard [11] extended Tanaka's work to non-Maxwellian molecules, see the last part of paper by Fournier [10] for up-to-date results.

In the case of long-range interactions, that is when particles interact through a repulsive force in $1/r^s$ (for some $s > 2$), the Boltzmann equation presents a singular integral (case without cutoff). The reason is that the corresponding process $(V_t)_{t\geq 0}$ jumps infinitely often, i.e. the particle is subjected to infinitely many collisions, on each time interval. In some sense, it behaves, roughly, like a Lévy process.

The Hölder regularity of the sample paths of stochastic processes was first studied by Orey and Taylor [17] and Perkins [18], who showed that the fast and slow points of Brownian motion are located on random sets of times, and they showed that the sets of points with a given pointwise regularity have a fractal nature. Jaffard [15] showed that the sample paths of most Lévy processes are multifractal functions and he obtained their spectrum of singularities. This spectrum is almost surely deterministic: of course, the sets with a given pointwise regularity are extremely complicated, but their Hausdorff dimension is deterministic. Let us also mention the article by Balança [4], in which he extended the results (and simplified some proofs) of Jaffard [15].

What we expect here is that $(V_t)_{t\geq 0}$ should have the same spectrum as a well-chosen Lévy process. This is of course very natural (having a look at the shape of the jumping SDE satisfied by $(V_t)_{t\geq 0}$). There are however many complications, compared to the case of Lévy processes, since we lose all the independence and stationarity properties that simplify many computations and arguments. We will also compute the multifractal spectrum of the position process $(X_t)_{t\geq 0}$, defined by $X_t = \int_0^t V_s ds$, which appears to have multifractal sample paths as well.

By the way, let us mention that, though there are many papers computing the multifractal spectrum of some quite complicated objects, we are not aware of any work concerning general Markov processes, that is, roughly, solutions to jumping (or even non jumping) SDEs. In this paper, we study the important case of the Boltzmann process, as a physical example of jumping SDE. Of course, a number of difficulties have to be surmounted, since the model is rather complicated. However, we follow, adapting everywhere to our situation, the main ideas of Jaffard [15] and Balança [4].

Let us finally mention that Barral, Fournier, Jaffard and Seuret [5] studied a very specific ad-hoc Markov process, showing that quite simple processes may have a random spectrum that depends heavily on the values taken by the process.

1.1. The Boltzmann equation

We consider a 3-dimensional spatially homogeneous Boltzmann equation, which depicts the density $f_t(v)$ of particles in a gas, moving with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The density $f_t(v)$ solves

$$\partial_t f_t(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma\, B(|v - v_*|, \cos\theta)[f_t(v') f_t(v'_*) - f_t(v) f_t(v_*)], \tag{1.1}$$

where

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \quad \text{and} \quad \cos\theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle. \tag{1.2}$$

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