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Evolutionary games on the torus with weak selection

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Abstract

We study evolutionary games on the torus with N points in dimensions $d \geq 3$. The matrices have the form $\bar{G} = \mathbf{1} + wG$, where $\mathbf{1}$ is a matrix that consists of all 1's, and w is small. As in Cox Durrett and Perkins (2011) we rescale time and space and take a limit as $N \to \infty$ and $w \to 0$. If (i) $w \gg N^{-2/d}$ then the limit is a PDE on $\mathbb{R}^d$. If (ii) $N^{-2/d} \gg w \gg N^{-1}$, then the limit is an ODE. If (iii) $w \ll N^{-1}$ then the effect of selection vanishes in the limit. In regime (ii) if we introduce mutations at rate μ so that $\mu/w \to \infty$ slowly enough then we arrive at Tarnita's formula that describes how the equilibrium frequencies are shifted due to selection.

Keywords: Voter model; Voter model perturbation; PDE limit; Tarnita's formula

1. Introduction

Here we will be interested in n-strategy evolutionary games on the torus $\mathbb{T}_L = (\mathbb{Z} \bmod L)^d$. Throughout the paper we will suppose that $n \geq 2$ and $d \geq 3$. The dynamics are described by a game matrix $G_{i,j}$ that gives the payoff to a player who plays strategy i against an opponent who plays strategy j. As in [7,8], we will study games with matrices of the form $\bar{G} = \mathbf{1} + wG$, and $\mathbf{1}$ is a matrix that consists of all 1's, and $w = \epsilon^2$. We use two notations for the small parameter to make it easier to connect with the literature.

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There are two commonly used update rules. To define them introduce

Assumption 1. Let p be a probability distribution on $\mathbb{Z}^d$ with finite range, $p(0) = 0$ and that satisfies the following symmetry assumptions.

- If π is a permutation of $\{1, 2, \dots, d\}$ and $(\pi z)_i = z_{\pi(i)}$ then $p(\pi z) = p(z)$.
- If we let $\hat{z}^i_i = -z_i$ and $\hat{z}^i_j = z_j$ for $j \neq i$ then $p(\hat{z}^i) = p(z)$.

For example, if $p(z) = f(\|z\|_p)$ where $\|z\|_p$ is the L^p norm on $\mathbb{Z}^d$ with $1 \le p \le \infty$ then the symmetry assumptions are satisfied.

Birth–Death Dynamics. In this version of the model, a site x gives birth at a rate equal to its fitness

$$\psi(x) = \sum_y p(y-x)\bar{G}(\xi(x), \xi(y))$$

and the offspring, which uses the same strategy as the parent, replaces a "randomly chosen neighbor of x". Here, and in what follows, the phrase in quotes means z is chosen with probability $p(z-x)$. Note that we use the same transition probability to compute the fitness and do the displacement. In general they can be different.

Death–Birth Dynamics. In this case, each site x dies at rate 1 and is replaced by the offspring of a neighbor y chosen with probability proportional to $p(y-x)\psi(y)$.

Tarnita et al. [23,24] have studied the behavior of evolutionary games on more general graphs when $w = o(1/N)$ and N is the number of vertices. To describe their results, we begin with the two strategy game written as

$$G = \begin{array}{ccc} & 1 & 2 \\ 1 & \alpha & \beta \\ 2 & \gamma & \delta. \end{array} \tag{1}$$

In [23] strategy 1 is said to be favored by selection (written $1 > 2$) if the frequency of 1 in equilibrium is $>1/2$ when w is small. Assuming that

(i) the transition probabilities are differentiable at $w = 0$,

(ii) the update rule is symmetric for the two strategies, and

(iii) strategy 1 is not disfavored in the game given with $\beta = 1$ and $\alpha = \gamma = \delta = 0$

they argued that

I. $1 > 2$ *is equivalent to* $\sigma\alpha + \beta > \gamma + \sigma\delta$ *where* σ *is a constant that only depends on the spatial structure and update rule.*

In [8] it was shown that for games on $\mathbb{Z}^d$ with $d \ge 3$.

Theorem 1. *I holds for the Birth–Death updating with* $\sigma = 1$ *and for the Death–Birth updating with* $\sigma = (\kappa+1)/(\kappa-1)$ *where*

$$\kappa = 1 \Big/ \sum_x p(x)p(-x) \tag{2}$$

is the effective number of neighbors.