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Microstructure noise in the continuous case: Approximate efficiency of the adaptive pre-averaging method

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Abstract

This paper introduces adaptiveness to the non-parametric estimation of volatility in high frequency data. We consider general continuous Itô processes contaminated by microstructure noise. In the context of pre-averaging, we show that this device gives rise to estimators that are within 7% of the commonly conjectured "quasi-lower bound" for asymptotic efficiency. The asymptotic variance is of the form constant × bound, where the constant does not depend on the process to be estimated. The results hold with mild assumptions on the noise, and extend to mildly irregular observations.

Keywords: Adaptive estimation; Discrete observation; Efficiency; High frequency data; Itô process; Leverage effect; Microstructure; Pre-averaging; Realized volatility; Semi-martingale; Stable convergence

1. Introduction

This paper is concerned with the estimation of integrated volatility for a one-dimensional Brownian semimartingale of the form

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s, \tag{1.1}$$

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so the quantity of interest is $C_T = \int_0^T \sigma_s^2\, ds$, when the underlying process is observed at regularly spaced times over the interval $[0, T]$,[1] and when it is contaminated with a so-called microstructure noise. The time span T here is *fixed*, whereas the inter-observations time Δ_n is small and in the asymptotic it goes to 0.

This topic has been the object of a large number of investigations in the past fifteen years, starting with the non-noisy case in the early papers [2,3,7,21,33]. The influence of microstructure noise has been considered, for example, in [5,35], for a white noise independent of the underlying process, and estimators with the efficient rate of convergence $n^{1/4}$ when n is the number of observations have been introduced in [34,6,29,18,32,30], with various assumptions on the noise.

We are concerned here with "asymptotic efficiency". Rate efficiency is achieved by many estimators in the previous references, but here we are interested in how low one can push the asymptotic variance. Genuine efficiency bounds, in the sense of Hajek convolution theorem for instance, is known when σ_t is time varying but non-random, or is a function of state variables, and noise is additive and Gaussian [30]. The variance bound involves σ and the noise variance. In more general situations concerning σ_t and/or the noise, it has been conjectured that the variance bound is the similar expression, but with the random σ_t plugged in. In a recent breakthrough paper, [1] finds an estimator based on the spectral approach of [30] which reaches the conjectured lower bound in a very general situation. For a compare and contrast, we refer to Remarks 3.1 and 5.2, where a more thorough discussion takes place.

Since several of the other approaches are widely applied, it is of some interest to see how close one can get to optimality for also for these methods. All methods necessitate the choice of a bandwidth and of a kernel or weight function, and for the other known approaches to volatility estimation, optimal estimators have not been found. For a given kernel, when $\sigma_t = \sigma$ is not varying one knows how to choose a bandwidth minimizing the asymptotic variance; in contrast, an optimal choice of the kernel is still an open problem, although some choices lead to a variance as low as 1.003 times the lower bound, and the very simple triangular kernel specified later gives a variance equal to 1.07 times the lower bound. On the other hand, for any given kernel a significant variability of σ_t induces the asymptotic variance to be *much* bigger than the (conjectured) efficient variance bound, if the bandwidth is taken to be the same over the whole time interval $[0, T]$; for example, with the variability of the spot volatility typically encountered within a day or a week, the effective asymptotic variance is as much as 4 or 5 times the lower bound.

The aim of this paper is to provide an adaptive device for the choice of the bandwidth, which allows us to obtain the minimal asymptotic variance associated with any particular choice of the kernel or weight function. We only consider the pre-averaging class [29,18] of estimators, although similar improvements can probably be done for other methods. We also emphasize that the method works for *all* weight functions, although in the Monte Carlo we illustrate it for the triangular function only. On the other hand, concerning the choice of the weight function, we only provide a heuristic discussion, see Sections 3 and 5. The reason is that in the pre-averaging setting an optimal choice may not even exist, although it is possible to achieve genuine efficiency by using optimized linear combinations of the estimators with a proper family of weight functions, such as the sine functions with various frequencies. The proof, however, would be much more involved and the practical implementation rather complicated. From a practical viewpoint, we feel that achieving an asymptotic variance smaller than 1.1 times the lower bound is "practically efficient", if not mathematically efficient.

[1] Though see Remark 4.4 at the end of Section 4.

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