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Comparison theorems for some backward stochastic Volterra integral equations☆

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Abstract

For some backward stochastic Volterra integral equations (BSVIEs) in multi-dimensional Euclidean spaces, comparison theorems are established in a systematic way for the adapted solutions and adapted M-solutions. For completeness, comparison theorems for (forward) stochastic differential equations, backward stochastic differential equations, and (forward) stochastic Volterra integral equations (FSVIEs) are also presented. Duality principles are used in some relevant proofs. Also, it is found that certain kinds of monotonicity conditions play crucial roles to guarantee the comparison theorems for FSVIEs and BSVIEs to be true. Various counterexamples show that the assumed conditions are almost necessary in some sense.

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1. Introduction

Throughout this paper, we let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which an m-dimensional standard Brownian motion $W(\cdot)$ is defined with $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ being its natural filtration augmented by all the $\mathbb{P}$-null sets. We consider the following equation in the usual

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n-dimensional real Euclidean space $\mathbb{R}^n$:

$$\begin{aligned} Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t))ds - \int_t^T Z(t, s)dW(s), \\ t \in [0, T], \end{aligned} \tag{1.1}$$

which is called a *backward stochastic Volterra integral equation* (BSVIE, for short). Such kind of equations have been investigated in the recent years (see [21,37–39,32,2], and references cited therein). BSVIEs are natural extensions of by now well-understood *backward stochastic differential equations* (BSDEs, for short) whose integral form is as follows:

$$Y(t) = \xi + \int_t^T g(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \quad t \in [0, T]. \tag{1.2}$$

See [27,8,23,41] for some standard results on BSDEs.

A pair of processes $(Y(\cdot), Z(\cdot\,,\cdot))$ is called an *adapted solution* to BSVIE (1.1) if for each $t \in [0, T]$, $s \mapsto (Y(s), Z(t, s))$ is $\mathbb{F}$-adapted on $[t, T]$ and (1.1) is satisfied in the usual Itô's sense. Further, an adapted solution $(Y(\cdot), Z(\cdot\,,\cdot))$ is called an *adapted M-solution* of (1.1) if, in addition, the following holds:

$$Y(t) = \mathbb{E}Y(t) + \int_0^t Z(t, s)dW(s), \quad t \in [0, T]. \tag{1.3}$$

In the case that the *generator* $g(\cdot)$ is independent of $Z(s, t)$, one can show that under proper conditions, BSVIE (1.1) admits a unique adapted solution $(t, s) \mapsto (Y(s), Z(t, s))$ with $0 \le t \le s \le T$. In this case, the values of $Z(t, s)$ for $0 \le s < t \le T$ are irrelevant. On the other hand, when the generator $g(\cdot)$ does depend on $Z(s, t)$, the adapted solution $(t, s) \mapsto (Y(s), Z(t, s))$ has to be defined on $[0, T] \times [0, T]$. In this case, adapted solution will not be unique in general. However, under suitable conditions, BSVIE (1.1) admits a unique adapted M-solution [39].

An interesting result of BSDEs is the comparison theorem for the adapted solutions. More precisely, for the case $n = 1$, if for $i = 0, 1$, $(Y^i(\cdot), Z^i(\cdot))$ is the adapted solution to the BSDE (1.2) with $(\xi, g(\cdot))$ replaced by $(\xi^i, g^i(\cdot))$ such that

$$\begin{cases} \xi^0 \le \xi^1, \quad \text{a.s.,} \\ g^0(t, y, z) \le g^1(t, y, z), \quad \forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}, \ \text{a.s.,} \end{cases} \tag{1.4}$$

then

$$Y^0(t) \le Y^1(t), \quad t \in [0, T], \ \text{a.s.} \tag{1.5}$$

The comparison theorem also holds for the case $n > 1$, see [15] for details. Thanks to the comparison theorem, the adapted solutions to BSDEs can be used as (time-consistent) dynamic risk measures or stochastic differential utility for (static) random variables which could be the payoff of a European type contingent claim at the maturity (see [8,28]).

Now, for BSVIEs, it is natural to ask if a comparison theorem similar to that for BSDEs holds. More precisely, if $(Y^i(\cdot), Z^i(\cdot\,,\cdot))$ is the adapted solution to BSVIE (1.1) when the generator $g(\cdot)$ is independent of $Z(s, t)$, or is the adapted M-solution of (1.1) when $g(\cdot)$ depends on $Z(s, t)$, with $(\psi(\cdot), g(\cdot))$ replaced by $(\psi^i(\cdot), g^i(\cdot))$, $i = 0, 1$, and

$$\begin{cases} \psi^0(t) \le \psi^1(t), \quad t \in [0, T], \ \text{a.s.,} \\ g^0(t, s, y, z, \zeta) \le g^1(t, s, y, z, \zeta), \quad 0 \le t \le s \le T, \ y, z, \zeta \in \mathbb{R}, \ \text{a.s.,} \end{cases} \tag{1.6}$$

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