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# Branching processes for the fragmentation equation

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**Abstract**

We investigate branching properties of the solution of a fragmentation equation for the mass distribution and we properly associate a continuous time càdlàg Markov process on the space $S^{\downarrow}$ of all fragmentation sizes, introduced by J. Bertoin. A binary fragmentation kernel induces a specific class of integral type branching kernels and taking as base process the solution of the initial fragmentation equation for the mass distribution, we construct a branching process corresponding to a rate of loss of mass greater than a given strictly positive threshold $d$. It turns out that this branching process takes values in the set of all finite configurations of sizes greater than $d$. The process on $S^{\downarrow}$ is then obtained by letting $d$ tend to zero. A key argument for the convergence of the branching processes is given by the Bochner–Kolmogorov theorem. The construction and the proof of the path regularity of the Markov processes are based on several newly developed potential theoretical tools, in terms of excessive functions and measures, compact Lyapunov functions, and some appropriate absorbing sets.


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## 1. Introduction

We study branching properties of the solution of a fragmentation equation for the mass distribution. Recall that the basic property of a measure-valued branching process is the following: if we consider two independent versions $X$ and $X'$ of the process, started respectively from two measures $\mu$ and $\mu'$, then $X + X'$ and the process started from $\mu + \mu'$ are equal in distribution. In studying the time evolution of fragmentation phenomena, it is supposed that "fragments split independently of each other", so, a branching property is fulfilled; cf. [5]. More specific, a main tool for defining the fragmentation chains is the branching Markov chains.

A different stochastic approach for studying the fragmentation (and coagulation) phenomena was developed in [22,23,29]: the evolution of the size of a typical particle in the system during a fragmentation process may be described by the solution of a stochastic differential equation, called *stochastic differential equation of fragmentation* (SDEF). It turns out that such a solution of the (SDEF) induces a solution of the fragmentation equation for the mass distribution.

In this paper we associate a continuous time Markov branching process to a fragmentation equation for the mass distribution, describing the time evolution of the fragments greater than a strictly positive size $d$. The model for the time evolution of all fragments (of arbitrary small size) is then constructed as a limit of a sequence of branching processes, corresponding to a fixed, decreasing sequence of thresholds $(d_n)_{n\geqslant 1}$ that converges to zero. This is a continuous time Markov process on the state $S^{\downarrow}$ of all fragmentation sizes, considered by J. Bertoin [5]. This process should be compared with the *stochastic coalescent process*, induced by Smoluchowski's coagulation equation in [41, page 95].

As a byproduct we emphasize integral type branching kernels on the space of all finite configurations of an interval $[d, 1]$, associated to the given fragmentation kernel and corresponding to the rate of loss of mass (in sense of [29]) greater than a fixed threshold $d$. These branching kernels lead to relevant examples of branching processes and it is possible to write down the nonlinear evolution equations satisfied by the associated cumulant semigroups.

The paper is organized as follows.

In the next section we present the fragmentation equation and the stochastic differential equation associated to it, following mainly [29]. A binary fragmentation kernel $F$ is fixed, we state some hypotheses, give the basic definitions, and an example. Proposition 2.2 shows that in the case when the fragmentation kernel $F$ is bounded, the existence and uniqueness of the solution of the fragmentation equation for the mass distribution holds and moreover, the existence of an associated Markov process with state space the interval $[0, 1]$, of the solution of the martingale problem, and of the weak solution of the (SDEF) also hold.

In Section 3 we first prove some properties of the real-valued Markov processes, produced by the procedure presented in Section 2, from the fragmentation kernel $F$ truncated to sizes greater than $d_n$. In particular, the interval $[d_n, d_{n-1})$ becomes an absorbing set and therefore it is possible to restrict the process to this set. Gathering together all these restrictions we obtain the base process of a forthcoming branching process on the set $\widehat{E_n}$ of all finite configurations of the set $E_n := [d_n, 1]$, $n \geqslant 1$. We show in Section 4 (Proposition 4.6) that the associated sequence of transition functions is a projective system and then, applying the Bochner–Kolmogorov theorem, we obtain a transition function on $S^{\downarrow}$ (Proposition 4.7). The already mentioned branching kernels associated to the given fragmentation kernel $F$, essential for constructing the branching processes, are also introduced in Section 4.

The results on the existence of the branching processes (with state spaces $\widehat{E_n}$, $n \geqslant 1$) and of the fragmentation process (with state space $S^{\downarrow}$) are proved in Section 5, Proposition 5.1