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Limit theorems for random walks that avoid bounded sets, with applications to the largest gap problem

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Abstract

Consider a centred random walk in dimension one with a positive finite variance σ^2, and let τ_B be the hitting time for a bounded Borel set B with a non-empty interior. We prove the asymptotic $\mathbb{P}_x(\tau_B > n) \sim \sqrt{2/\pi}\sigma^{-1} V_B(x) n^{-1/2}$ and provide an explicit formula for the limit V_B as a function of the initial position x of the walk. We also give a functional limit theorem for the walk conditioned to avoid B by the time n. As a main application, we consider the case that B is an interval and study the size of the largest gap G_n (maximal spacing) within the range of the walk by the time n. We prove a limit theorem for G_n, which is shown to be of the constant order, and describe its limit distribution. In addition, we prove an analogous result for the number of non-visited sites within the range of an integer-valued random walk.

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1. Introduction and results

1.1. Introduction

Let $S_n = x + X_1 + \cdots + X_n$ be a random walk in dimension one with centred increments that have a positive finite variance σ^2. Denote $\tau_B := \min\{k \geq 0 : S_k \in B\}$ the hitting time of a Borel

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set B. The goal of this paper is to find the asymptotic of $\mathbb{P}_x(\tau_B > n)$ as $n \to \infty$ in the case that B is bounded. Here $\mathbb{P}_x$ denotes the distribution of the walk starting at $S_0 = x$.

Recall that the distribution of X_1 is called λ-arithmetic if $\lambda > 0$ is the maximal number such that $\mathbb{P}(X_1 \in \lambda\mathbb{Z}) = 1$, and is called non-arithmetic if such a number does not exist. We set the state space M of the walk to be $\lambda\mathbb{Z}$ for λ-arithmetic walks and $\mathbb{R}$ for non-arithmetic walks. We will consider the starting points $x \in M$ only; we stress that this does not result in any loss of generality. To avoid trivialities, we *always* assume that $Int_M(B)$ is non-empty, where in the arithmetic case the interior is taken under the discrete topology.

Kesten and Spitzer [13] (Theorem 4(a)) proved that for *any* 1-arithmetic random walk, for any finite non-empty $B \subset \mathbb{Z}$ and $x \in \mathbb{Z}$ there exists the finite limit of the ratios

$$\lim_{n\to\infty} \frac{\mathbb{P}_x(\tilde{\tau}_B > n)}{\mathbb{P}_0(\tilde{\tau}_{\{0\}} > n)} =: g_B(x), \tag{1}$$

where $\tilde{\tau}_B := \min\{k \geq 1 : S_k \in B\}$ (of course $\tilde{\tau}_B = \tau_B$ for $x \notin B$). This is true no matter if the walk is transient or recurrent, and also holds in *dimension two* with the naturally extended the definition of 1-arithmeticity. Moreover, [13] gives a formula for the limit function, which under our assumptions reads as

$$g_B(x) = \lim_{y\to\infty} \frac{1}{2}\mathbb{E}_x \sum_{k=0}^{\tilde{\tau}_B} \mathbb{1}_{\{|X_k|=y\}}. \tag{2}$$

The proof of (1) in [13] is by induction in the number of elements in B. Essentially, the problem reduces to the case that $B = \{0\}$. It is studied by the means of potential theory for random walks, a probabilistic counterpart of the classical potential theory for the Laplace operator that was developed by Spitzer; for a detailed account see Chapters III and VIII of his book [18]. The asymptotic $\mathbb{P}_0(\tilde{\tau}_{\{0\}} > n)$ can be easily found using the theory of recurrent events. Thus (1) implies that under our assumptions,

$$\lim_{n\to\infty} \sqrt{n}\mathbb{P}_x(\tau_B > n) = \sqrt{\frac{2}{\pi}}\sigma a_B(x) \tag{3}$$

with $a_B(x) := g_B(x)\mathbb{1}_{B^c}(x),\ x \in \mathbb{Z}$. Similarly, the asymptotic of $\mathbb{P}_x(\tau_B > n)$ can be obtained for any asymptotically stable recurrent 1-arithmetic random walk, see Lemma 2.1 by Belkin [2].

It is worth to mention that the limit function $a_B(x)$ is harmonic for the Markov chain $\bar{S}_n := S_{n\wedge\tau_B}$, in the sense that $\mathbb{E}_x a_B(\bar{S}_1) = a_B(x)$ for any x. The chain $\bar{S}_n$ is usually referred to as the random walk killed as it enters B. One may also think that a_B is harmonic for the corresponding sub-stochastic transition kernel defined on B^c. On the other hand, it holds that $\mathbb{E}_x a_B(S_1) = a_B(x)$ for $x \notin B$ and $a_B(x) = 0$ for $x \in B$. Thus $a(x)$ is a non-negative solution of the exterior Dirichlet problem for the transition operator of the random walk with the boundary condition $a_B \equiv 0$ on B. Note that there is no uniqueness result for non-negative solutions of this problem for *general* random walks.

It is clear that the method of Kesten and Spitzer does not work in the general case of non-arithmetic walks, which is the main interest of this paper. We will use an entirely different approach that forces us to consider dimension one and assume the finiteness of variance. Although we believe that the proposed methodology provides an apt tool to study the problem for random walks with infinite variance, in the later case the fluctuation theory of ladder variables is totally different. This should result in a completely different proof while the approach of Kesten and Spitzer works for any random walk, including the ones that are asymptotically stable.