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# Growth rates of the population in a branching Brownian motion with an inhomogeneous breeding potential

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**Abstract**

We consider a branching particle system where each particle moves as an independent Brownian motion and breeds at a rate proportional to its distance from the origin raised to the power $p$, for $p \in [0, 2)$. The asymptotic behaviour of the right-most particle for this system is already known; in this article we give large deviations probabilities for particles following "difficult" paths, growth rates along "easy" paths, the total population growth rate, and we derive the optimal paths which particles must follow to achieve this growth rate.


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*Keywords:* Branching Brownian motion; Inhomogeneous breeding; Large deviations; Growth rate

## 1. Introduction and heuristics

### 1.1. The model

We study a branching Brownian motion (BBM) in an inhomogeneous breeding potential on $\mathbb{R}$. Fix $\beta > 0$, $p \in [0, 2)$, and a random variable $A$, which takes values in $\{1, 2, \ldots\}$, satisfying

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$\mathbb{E}[A \log A] < \infty$. We initialise our branching process with a single particle at the origin. Each particle $u$, once born, moves as a Brownian motion, independently of all other particles in the population. Each particle $u$ alive at time $T$ dies with instantaneous rate $\beta|X_u(T)|^p$, where $X_u(T)$ is the spatial position of particle $u$ (or of its ancestor) at time $T$. Upon death, a particle $u$ is replaced by a random number $1 + A_u$ of offspring in the same spatial position, where each $A_u$ is an independent copy of $A$. We define $m := \mathbb{E}[A]$, the average increase in the population size at each branching event. We denote by $N(T)$ the set of particles alive at time $T$. We let $\mathbb{P}$ represent the probability law, and $\mathbb{E}$ the corresponding expectation, of this BBM.

The case $p = 2$ is critical for this BBM: if the breeding rate were instead $\beta|\cdot|^p$ for $p > 2$, it is known from Itô and McKean [23, Sections 5.12 to 5.14] that the population explodes in finite time, almost surely. For $p = 2$, the *expected* number of particles explodes in finite time, but the population remains finite, almost surely, for all time.

Branching Brownian motions are closely associated with certain partial differential equations. In particular, for the above BBM model, the McKean representation tells us that

$$v(T, x) := \mathbb{E}\left(\prod_{u \in N(T)} f(x + X_u(T))\right)$$

solves the equation

$$\frac{\partial v}{\partial T} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \beta|x|^p(G(v) - v) \tag{1}$$

with the initial condition $v(0, x) = f(x)$, where $G(s) := \mathbb{E}(s^A)$ is the generating function of the offspring distribution $A$. In the case of constant branching rate ($p = 0$), this is known as the Fisher–Kolmogorov–Piscounov–Petrovski (FKPP) reaction–diffusion equation.

An object of fundamental importance in the study of branching diffusions is the right-most particle, defined as $R_T := \max_{u \in N(T)} X_u(T)$. Standard BBM, with binary branching at a constant rate (that is, $p = 0$ and $G(s) = s^2$), has been much studied. In this case, it is well known that the linear asymptotic $\lim_{T\to\infty} R_T/T = \sqrt{2\beta}$ holds almost surely. The distribution function of the right most particle position solves the FKPP equation with Heaviside initial conditions, and it is known that $\mathbb{P}(R_T \geq m(T) + x) \to w(x)$ where $w$ is a travelling-wave solution of (1) and $m(T)$ is the median for the rightmost particle position at time $T$. Sub-linear terms for the asymptotic behaviour of $m(T) = \sqrt{2\beta}T - 3/(2\sqrt{2\beta}) \log T + O(1)$ were found by Bramson [7] and [8]. See also the recent shorter probabilistic proofs by Roberts [31], and corresponding results for branching random walk by Aidekon [1] and Hu and Shi [22]. For approaches using partial differential equation theory, see the recent short proof by Hamel et al. [15] and an impressive higher order expansion due to Van Saarloos [32]. Detailed studies of the paths followed by the right-most particles have been carried out by Arguin et al. [3,4], and by Aidekon et al. [2].

For $p \in (0, 2)$, right most particle speeds much faster than linear occur and Harris and Harris [19] found an asymptotic for $R_T$ using probabilistic techniques involving additive martingales and changes of measure.

**Theorem 1** (*Harris, Harris [19]*). *For $p \in [0, 2)$,*

$$\lim_{T\to\infty} \frac{R_T}{T^{\frac{2}{2-p}}} = \left(\frac{m\beta}{2}(2-p)^2\right)^{\frac{1}{2-p}}$$

*almost surely.*