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Statistical and renewal results for the random sequential adsorption model applied to a unidirectional multicracking problem

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Abstract

We work out a stationary process on the real line to represent the positions of the multiple cracks which are observed in some composites materials submitted to a fixed unidirectional stress ε. Our model is the one-dimensional random sequential adsorption. We calculate the intensity of the process and the distribution of the inter-crack distance in the Palm sense. Moreover, the successive crack positions of the one-sided process (denoted by X_i^ε, $i \geqslant 1$) are described. We prove that the sequence $\{(X_i^\varepsilon, Y_i^\varepsilon), 1 \leqslant i \leqslant n\}$ is a "conditional renewal process", where Y_i^ε is the value of the stress at which X_i^ε forms. The approaches "in the Palm sense" and "one-sided process" merge when $n \to +\infty$. The saturation case ($\varepsilon = +\infty$) is also investigated.

MSC: primary 60G10; 60G55; 60K40; secondary 60B10; 60J05; 60K15

Keywords: Brittleness; Car-parking model; Coating; Composite material; Crack; Fibre; Markov chain; Packing model; Palm measure; Poisson point process; Random sequential adsorption; Relaxation of stress; Renewal process; Rupture; Stationary processes

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0. Introduction

Let $\{X_i; i \geqslant 1\}$ be a sequence of independent and uniformly distributed variables on the segment $[0, L]$, $L > 0$. We throw successively $X_1, \ldots, X_N$ on this segment, keeping only some of them according to the following procedure. For $N \geqslant 2$, we keep X_1 and after that, we erase X_2 if and only if X_2 is in the interval of radius $r > 0$ around X_1. Once we have decided if $X_2, \ldots, X_n$, $2 \leqslant n \leqslant (N-1)$ are kept or not, we erase X_{n+1} if it belongs to the union of all the intervals centered on the non-erased points, with length $2r$.

This construction is known as the one-dimensional random sequential adsorption (RSA) [3,2]. In spite of its simplicity, this model is difficult to deal with: in particular, the law of the number of preserved points is unknown.

In 1958, Rényi [18] worked out a model where the points are placed on the segment up to saturation (i.e. when no more point can be added). He obtained the asymptotic behaviour of the mean number of points in $[0, L]$ when L goes to infinity. This question, known as the car-parking problem, has been largely investigated (see for example [4,5,11,16,17]).

In 1966, fixing the number N of thrown variables, Widom [21] demonstrated by heuristic methods that the mean number of points which are separated from their right-neighbor by a fixed length $l > 0$ satisfies a differential equation in l. Moreover, he provided formulas for the empirical distribution function of the inter-point distance when $N, L \to +\infty$, with N/L fixed.

In this paper, we are interested mainly in modelling a unidirectional multicracking phenomenon of brittle coatings. A uniaxial strain is applied to a specimen consisting of a ductile substrate covered with a brittle coating. The applied strain is supposed to result in the coating in a regularly increasing stress denoted by ε, which leads to the formation of cracks parallel and orthogonal to the stress direction [7,12]. Consequently, the geometrical aspect of the problem reduces practically to the intersections of the cracks with a fixed line parallel with the stress axis. It has been observed [1,7,10,12] that the formation of a crack in the coating results in a relaxation of the stress in the vicinity of this crack so that no new crack can form close to an existing crack because of the smallness of the stress in this area. Consequently, the above-described RSA construction can be considered as a model for the crack positions.

More precisely, we construct through the RSA procedure a one-dimensional stationary point process Λ_ε, that represents the positions of cracks for a fixed value of the applied stress $\varepsilon > 0$. The parameter ε plays a central role in the model. In particular, the limit $\varepsilon \to +\infty$ corresponds to saturation.

The first section of our paper is devoted to the construction of Λ_ε. We start with a two-dimensional Poisson process Φ on $\mathbb{R} \times \mathbb{R}_+$ of intensity measure $\mathbf{1}_{\mathbb{R}_+}(y) f(y)\, \mathrm{d}x\, \mathrm{d}y$. In the physics literature, f is called (see e.g. [13]) rupture probability density of the coating. It is a non-decreasing function and therefore expresses the fact that the number of cracks grows with stress. From a mathematical point of view, there is no loss of generality in assuming that $f = 1$ (see the beginning of Section 1 for details).