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Low regularity solutions to a gently stochastic nonlinear wave equation in nonequilibrium statistical mechanics

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Abstract

We consider a system of stochastic partial differential equations modeling heat conduction in a non-linear medium. We show global existence of solutions for the system in Sobolev spaces of low regularity, including spaces with norm beneath the energy norm. For the special case of thermal equilibrium, we also show the existence of an invariant measure (Gibbs state).
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1. Introduction

In this article we consider the following system of partial differential

$$\begin{aligned}
&\partial_t \phi(x,t) = \pi(x,t),\\
&\partial_t \pi(x,t) = (\partial_x^2 - 1)\phi(x,t) - \mu\phi^3(x,t) - r(t)\alpha(x),\\
&\mathrm{d}r(t) = -(r(t) - \langle \alpha, \pi(t)\rangle)\,\mathrm{d}t + \sqrt{2T}\,\mathrm{d}\omega(t). \qquad (1)
\end{aligned}$$

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In Eqs. (1) (ϕ, π) is a pair of scalar fields satisfying periodic boundary conditions with $x \in [0, 2\pi]$. The vector-valued functions $\alpha = (\alpha_1, \dots, \alpha_K)$ has each component $\alpha_i(x)$ in the Sobolev space H^γ for some $\gamma > 0$. The vector $r(t) = (r_1(t), \dots, r_K(t))$ takes value in $\mathbf{R}^K$. Here $r(t)\alpha(x) = \sum_{i=1}^K r_i(t)\alpha_i(x)$ and $\langle \alpha, \pi(t) \rangle$ is the vector with values in $\mathbf{R}^K$ and with components $\langle \alpha_i, \pi(t) \rangle$ where $\langle \cdot, \cdot \rangle$ is the $L^2([0, 2\pi])$ inner product. Finally $\omega(t) = (\omega_1(t), \dots, \omega_K(t))$ is a standard K-dimensional Brownian motion, and $\sqrt{2T}\, \mathrm{d}\omega$ has components $\sqrt{2T_i}\, \mathrm{d}\omega_i$ and T_i is interpreted as a temperature. The parameter μ is a coupling constant; we will be primarily interested in the cases $\mu = 0$ (linear Klein–Gordon equation) and $\mu > 0$ (non-linear defocusing linear wave equation).

The system of equations (1) arises from a model for heat conduction in a nonlinear medium. It can be derived from first principles from a Hamiltonian system which consists of K linear wave equations in $\mathbf{R}$ coupled to a nonlinear wave equation in $[0, 2\pi]$. The total Hamiltonian is given by

$$
\begin{aligned}
H = {} & \sum_{j=1}^K \int_{\mathbf{R}} \frac{1}{2} (|\partial_x u_j(x)|^2 + |v_j(x)|^2)\, \mathrm{d}x \\
& + \int_{[0,2\pi]} \frac{1}{2} (|\partial_x \phi(x)|^2 + |\phi(x)|^2 + |\pi(x)|^2) + \frac{\mu}{4} |\phi(x)|^4\, \mathrm{d}x \\
& + \sum_{j=1}^K \left(\int_{\mathbf{R}} \partial_x u_j(x) \rho_j(x)\, \mathrm{d}x \right) \left(\int_{[0,2\pi]} \partial_x \phi(x) \alpha_j(x)\, \mathrm{d}x \right),
\end{aligned}
\tag{2}
$$

with the ρ_j's and the α_j's fixed coupling functions. One assumes further that the initial conditions of the (u_j, v_j), $j = 1, \dots, K$ ("the reservoirs") are distributed according to Gibbs measures at temperatures T_j. These measures are (formally) expressed as

$$
Z^{-1} \exp\left(-\frac{1}{2T_i} \int_{\mathbf{R}} (|\partial_x u_j(x)|^2 + |v_j(x)|^2)\, \mathrm{d}x \right) \prod_{x \in \mathbf{R}} \mathrm{d}u_j(x)\, \mathrm{d}v_j(x),
\tag{3}
$$

and they are simply the product of a Wiener measure (for the position fields u_j) with a white noise measure (for the momenta fields v_j).

We refer to [11] or [20,18] for details on the derivation of equations (1) from the Hamiltonian system (2) with initial conditions (3), at least in the case where the nonlinear wave equation is replaced by a chain of nonlinear oscillators (formally a discrete wave equation). In that case one obtains a set of stochastic ordinary differential equations. The derivation is essentially the same as for the model considered here and will not be repeated. We simply remark that the derivation of Markovian equations is possible due to a particular choice of the ρ_j's.

In a series of papers [11,12,19–21,8,10,18] about the chain of nonlinear oscillators, the existence, uniqueness, and strong ergodic properties of invariant measures have been established. Moreover, a number of properties of these invariant measures have been elucidated, such as existence of heat flow, positivity of entropy production, and symmetry properties of entropy production fluctuations. These invariant measures represent stationary states which generalize Gibbs distributions to non-equilibrium situations where there is heat flow. Ultimately our goal is to establish similar