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# A general study of extremes of stationary tessellations with examples

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## Abstract

Let $\mathfrak{m}$ be a random tessellation in $\mathbf{R}^d$, $d \geq 1$, observed in a bounded Borel subset $W$ and $f(\cdot)$ be a measurable function defined on the set of convex bodies. A point $z(C)$, called the nucleus of $C$, is associated with each cell $C$ of $\mathfrak{m}$. Applying $f(\cdot)$ to all the cells of $\mathfrak{m}$, we investigate the order statistics of $f(C)$ over all cells $C \in \mathfrak{m}$ with nucleus in $\mathbf{W}_\rho = \rho^{1/d} W$ when $\rho$ goes to infinity. Under a strong mixing property and a local condition on $\mathfrak{m}$ and $f(\cdot)$, we show a general theorem which reduces the study of the order statistics to the random variable $f(\mathcal{C})$, where $\mathcal{C}$ is the typical cell of $\mathfrak{m}$. The proof is deduced from a Poisson approximation on a dependency graph via the Chen–Stein method. We obtain that the point process $\left\{(\rho^{-1/d} z(C), a_\rho^{-1}(f(C) - b_\rho)), C \in \mathfrak{m}, z(C) \in \mathbf{W}_\rho\right\}$, where $a_\rho > 0$ and $b_\rho$ are two suitable functions depending on $\rho$, converges to a non-homogeneous Poisson point process. Several applications of the general theorem are derived in the particular setting of Poisson–Voronoi and Poisson–Delaunay tessellations and for different functions $f(\cdot)$ such as the inradius, the circumradius, the area, the volume of the Voronoi flower and the distance to the farthest neighbor.



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## 1. Introduction

A tessellation of $\mathbf{R}^d$, $d \geq 1$, endowed with its Euclidean norm $|\cdot|$, is a countable collection of nonempty compact subsets, called *cells*, with disjoint interiors which subdivides the space and such that the number of cells intersecting any bounded subset of $\mathbf{R}^d$ is finite. The set $\mathbf{T}$ of tessellations is endowed with the $\sigma$-algebra generated by the sets $\left\{m \in \mathbf{T}, \bigcup_{C \in m} \partial C \cap K = \varnothing\right\}$ where $\partial C$ is the boundary of $K$ for any compact set $C$ in $\mathbf{R}^d$. By a random tessellation $\mathfrak{m}$, we mean a random variable with values in $\mathbf{T}$. It is said to be stationary if its distribution is invariant under translations of the cells. For a complete account on random tessellations, we refer to the books [35,40] and the survey [6].

Given a fixed realization of $\mathfrak{m}$, we associate with each cell $C \in \mathfrak{m}$ in a deterministic way a point $z(C)$, which is called the *nucleus* of the cell, such that $z(C + x) = z(C) + x$ for all $x \in \mathbf{R}^d$. To describe the mean behavior of the tessellation, the notions of intensity and typical cell are introduced as follows. Let $B$ be a Borel subset of $\mathbf{R}^d$ such that $\lambda_d(B) \in (0, \infty)$, where $\lambda_d$ is the $d$-dimensional Lebesgue measure. The *intensity* $\gamma$ of the tessellation is defined as $\gamma = \frac{1}{\lambda_d(B)} \cdot \mathbb{E}\left[\#\{C \in \mathfrak{m}, z(C) \in B\}\right]$ and we assume that $\gamma \in (0, \infty)$. Since $\mathfrak{m}$ is stationary, $\gamma$ is independent of $B$ and we suppose, without loss of generality, that $\gamma = 1$. The *typical cell* $\mathcal{C}$ is a random polytope whose distribution is given by

$$\mathbb{E}[f(\mathcal{C})] = \frac{1}{\lambda_d(B)} \cdot \mathbb{E}\left[\sum_{\substack{C \in \mathfrak{m}, \\ z(C) \in B}} f(C - z(C))\right] \tag{1.1}$$

for all $f : \mathcal{K}_d \to \mathbf{R}$ bounded measurable functions on the set of convex bodies $\mathcal{K}_d$, i.e. convex compact sets, endowed with the Hausdorff topology.

We are interested in the following problem: only a part of the tessellation is observed in the window $\mathbf{W}_\rho = \rho^{1/d} W$, where $W$ is a bounded Borel subset of $\mathbf{R}^d$, i.e. included in a cube $\mathbf{C}^{(W)}$, and such that $\lambda_d(W) \neq 0$. Let $f : \mathcal{K}_d \to \mathbf{R}$ be a translation invariant measurable function, i.e. $f(C + x) = f(C)$ for all $C \in \mathcal{K}_d$ and $x \in \mathbf{R}^d$. For all $r \in \mathbf{N}^*$, we denote by $M^{(r)}_{f,\mathbf{W}_\rho}$ the $r$th order statistic of $f$ over the cells $C \in \mathfrak{m}$ such that $z(C) \in \mathbf{W}_\rho$. We have chosen the convention to call the $r$ order statistics the $r$ largest values. When $r = 1$, the 1st order statistic is denoted by $M_{f,\mathbf{W}_\rho}$, i.e.

$$M_{f,\mathbf{W}_\rho} = M^{(1)}_{f,\mathbf{W}_\rho} = \max_{\substack{C \in \mathfrak{m}, \\ z(C) \in \mathbf{W}_\rho}} f(C).$$

In this paper, we investigate the limit behavior of $M^{(r)}_{f,\mathbf{W}_\rho}$ when $\rho$ goes to infinity.

The study of extremes could describe the regularity of the tessellation (e.g. presence of elongated cells). For instance, in the finite element method, the quality of the approximation depends on some consistency measurements over the partition, see e.g. [14]. Another potential application field is statistics of point processes. The key idea would be to identify a point process from the extremes of a tessellation induced by the point process.

To the best of our knowledge, one of the first works on extreme values in stochastic geometry is due to Penrose. In Chapters 6, 7 and 8 in [26], he investigates the maximum and minimum degrees of random geometric graphs. More recently, Schulte and Thäle [36] establish a theorem to derive the order statistics of a functional $f_k(x_1, \ldots, x_k)$ of $k$ points on a Poisson point process. Nevertheless, their approach cannot be applied to our problem. Indeed, studying ex-