SPA: 2593 Model 1 pp. 1–22 (col. fig: NIL)

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Stochastic Processes and their Applications xx (xxxx) xxx–xxx

stochastic processes and their applications

www.elsevier.com/locate/spa

Generalized Gaussian bridges

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Received 16 April 2013; received in revised form 3 April 2014; accepted 4 April 2014

Abstract

A generalized bridge is a stochastic process that is conditioned on N linear functionals of its path. We consider two types of representations: orthogonal and canonical. The orthogonal representation is constructed from the entire path of the process. Thus, the future knowledge of the path is needed. In the canonical representation the filtrations of the bridge and the underlying process coincide. The canonical representation is provided for prediction-invertible Gaussian processes. All martingales are trivially prediction-invertible. A typical non-semimartingale example of a prediction-invertible Gaussian process is the fractional Brownian motion. We apply the canonical bridges to insider trading.

MSC: 60G15; 60G22; 91G80

Keywords: Canonical representation; Enlargement of filtration; Fractional Brownian motion; Gaussian process; Gaussian bridge; Hitsuda representation; Insider trading; Orthogonal representation; Prediction-invertible process; Volterra process

1. Introduction

Let $X = (X_t)_{t\in[0,T]}$ be a continuous Gaussian process with positive definite covariance function R, mean function m of bounded variation, and $X_0 = m(0)$. We consider the conditioning, or bridging, of X on N linear functionals $\mathbf{G}_T = [G^i_T]_{i=1}^N$ of its paths:

$$\mathbf{G}_T(X) = \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \left[\int_0^T g_i(t)\,\mathrm{d}X_t\right]_{i=1}^N. \tag{1.1}$$

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http://dx.doi.org/10.1016/j.spa.2014.04.002

0304-4149/© 2014 Published by Elsevier B.V.

We assume, without any loss of generality, that the functions g_i are linearly independent. Indeed, if this is not the case then the linearly dependent, or redundant, components of $\mathbf{g}$ can simply be removed from the conditioning (1.2) without changing it.

The integrals in the conditioning (1.1) are the so-called abstract Wiener integrals (see Definition 2.5 later). The abstract Wiener integral $\int_0^T g(t)\,\mathrm{d}X_t$ will be well-defined for functions or generalized functions g that can be approximated by step functions in the inner product $\langle\!\langle\!\langle \cdot, \cdot \rangle\!\rangle\!\rangle$ defined by the covariance R of X by bilinearly extending the relation $\langle\!\langle\!\langle 1_{[0,t)}, 1_{[0,s)} \rangle\!\rangle\!\rangle = R(t,s)$. This means that the integrands g are equivalence classes of Cauchy sequences of step functions in the norm $|\!|\!| \cdot |\!|\!|$ induced by the inner product $\langle\!\langle\!\langle \cdot, \cdot \rangle\!\rangle\!\rangle$. Recall that for the case of Brownian motion we have $R(t,s) = t \wedge s$. Therefore, for the Brownian motion, the equivalence classes of step functions are simply the space $L^2([0,T])$.

Informally, the generalized Gaussian bridge $X^{\mathbf{g};\mathbf{y}}$ is (the law of) the Gaussian process X conditioned on the set

$$\left\{ \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \mathbf{y} \right\} = \bigcap_{i=1}^N \left\{ \int_0^T g_i(t)\,\mathrm{d}X_t = y_i \right\}. \tag{1.2}$$

The rigorous definition is given in Definition 1.3 later.

For the sake of convenience, we will work on the canonical filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\Omega = C([0,T])$, $\mathscr{F}$ is the Borel σ-algebra on $C([0,T])$ with respect to the supremum norm, and $\mathbb{P}$ is the Gaussian measure corresponding to the Gaussian coordinate process $X_t(\omega) = \omega(t)$: $\mathbb{P} = \mathbb{P}[X \in \cdot\,]$. The filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ is the intrinsic filtration of the coordinate process X that is augmented with the null-sets and made right-continuous.

Definition 1.3. The *generalized bridge measure* $\mathbb{P}^{\mathbf{g};\mathbf{y}}$ is the regular conditional law

$$\mathbb{P}^{\mathbf{g};\mathbf{y}} = \mathbb{P}^{\mathbf{g};\mathbf{y}}[X \in \cdot\,] = \mathbb{P}\left[X \in \cdot \;\middle|\; \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \mathbf{y} \right].$$

A *representation of the generalized Gaussian bridge* is any process $X^{\mathbf{g};\mathbf{y}}$ satisfying

$$\mathbb{P}\left[X^{\mathbf{g};\mathbf{y}} \in \cdot\, \right] = \mathbb{P}^{\mathbf{g};\mathbf{y}}[X \in \cdot\,] = \mathbb{P}\left[X \in \cdot \;\middle|\; \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \mathbf{y} \right].$$

Note that the conditioning on the $\mathbb{P}$-null-set (1.2) in Definition 1.3 is not a problem, since the canonical space of continuous processes is a Polish space and all Polish spaces are Borel spaces and thus admit regular conditional laws, cf. [20, Theorems A1.2 and 6.3]. Also, note that as a *measure* $\mathbb{P}^{\mathbf{g};\mathbf{y}}$ the generalized Gaussian bridge is unique, but it has several different *representations* $X^{\mathbf{g};\mathbf{y}}$. Indeed, for any representation of the bridge one can combine it with any $\mathbb{P}$-measure-preserving transformation to get a new representation.

In this paper we provide two different representations for $X^{\mathbf{g};\mathbf{y}}$. The first representation, given by Theorem 3.1, is called the *orthogonal representation*. This representation is a simple consequence of orthogonal decompositions of Hilbert spaces associated with Gaussian processes and it can be constructed for any continuous Gaussian process for any conditioning functionals. The second representation, given by Theorem 4.25, is called the *canonical representation*. This representation is more interesting but also requires more assumptions. The canonical representation is dynamically invertible in the sense that the linear spaces $\mathscr{L}_t(X)$ and $\mathscr{L}_t(X^{\mathbf{g};\mathbf{y}})$ (see Definition 2.1 later) generated by the process X and its bridge representation $X^{\mathbf{g};\mathbf{y}}$ coincide for all times $t \in [0,T)$. This means that at every time point $t \in [0,T)$ the bridge and