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An excursion approach to maxima of the Brownian bridge

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Abstract

Distributions of functionals of Brownian bridge arise as limiting distributions in non-parametric statistics. In this paper we will give a derivation of distributions of extrema of the Brownian bridge based on excursion theory for Brownian motion. The idea of rescaling and conditioning on the local time has been used widely in the literature. In this paper it is used to give a unified derivation of a number of known distributions, and a few new ones. Particular cases of calculations include the distribution of the Kolmogorov–Smirnov statistic and the Kuiper statistic.

MSC: 60J65; 60G55; 60J55

Keywords: Brownian bridge; Rescaling; Excursions; Extrema; Kolmogorov–Smirnov statistics

1. Introduction

Distributions of functionals of Brownian bridge arise as limiting distributions in non-parametric statistics. The distribution of the maximum of the absolute value of a Brownian bridge

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is the basis for the Kolmogorov–Smirnov non-parametric test of goodness of fit to give one example. For an overview of statistical applications see [23].

Let $(\mathbb{U}_t : 0 \le t \le 1)$ be the standard Brownian bridge and define

$$M^+ = \max_{0 \le t \le 1} \mathbb{U}_t, \qquad M^- = -\min_{0 \le t \le 1} \mathbb{U}_t \tag{1.1}$$

and

$$m = \min\{M^+, M^-\} \quad \text{and} \quad M = \max\{M^+, M^-\}. \tag{1.2}$$

The distribution of M^+ was first computed by Smirnov (1939). The derivation of the distribution of M was given by Kolmogorov [12]. For an elementary derivation see [9]. In this paper we give a derivation of the joint distribution of M^+ and M^- based on excursion theory for Brownian motion. The distributional results can be used to derive known distributions like the distribution of the Kuiper statistic $K = M^+ + M^-$, or the distribution of the difference $D = M^+ - M^-$ which seems to be new.

Let $(B_t : t \ge 0)$ be standard Brownian motion. Define the last exit time from 0 of B before time t as

$$g_t = \sup\{s \le t : B_s = 0\}. \tag{1.3}$$

The following lemma is well known and will be used to derive distributional equalities needed later. See [15,7,1].

Theorem 1. *The distribution of g_1 is* Beta$(1/2, 1/2)$. *Given g_1, the process $(B_t : 0 \le t \le g_1)$ is a Brownian bridge of length g_1, and the rescaled process $(B(g_1 u)/\sqrt{g_1} : 0 \le u \le 1)$ is a Brownian bridge independent of g_1.*

Let $S_\theta \sim \exp(\theta)$ be independent from B. From scaling properties of Brownian motion and Theorem 1 it follows that the process

$$\left(\frac{B_{t g_{S_\theta}}}{\sqrt{g_{S_\theta}}} : 0 \le t \le 1 \right) \tag{1.4}$$

is a Brownian bridge independent of g_{S_θ}. Furthermore, the law of g_{S_θ} is equal to the law of $g_1 S_\theta$ where g_1 and S_θ are assumed to be independent which is known to be $\Gamma(1/2, \theta)$; here $X \sim \Gamma(a, b)$ means that X has the Gamma density

$$p(x; a, b) = \frac{b(bx)^{a-1}}{\Gamma(a)} \exp(-bx) 1_{(0,\infty)}(x).$$

Let $\mathbb{U}$ be the standard Brownian bridge. Let $\gamma \sim \Gamma(1/2, \theta)$ be independent from $\mathbb{U}$ and let

$$\tilde{\mathbb{U}}_t = \sqrt{\gamma}\, \mathbb{U}_{t/\gamma} \tag{1.5}$$

for $0 \le t \le \gamma$. The process $(\tilde{\mathbb{U}}_t : 0 \le t \le \gamma)$ is called the randomly rescaled Brownian bridge. From the independence of g_{S_θ} and the process defined in (1.4) we have

$$(\tilde{\mathbb{U}}_t : 0 \le t \le \gamma) \stackrel{\mathrm{d}}{=} \left(B_t : 0 \le t \le g_{S_\theta}\right). \tag{1.6}$$

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